NONSYMBOLIC LOGIC (BRIEFLY and PARTIALLY)

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SOME SECTIONS CONSIST OF DEVELOPMENT NOTES. MOST SECTIONS ARE PARTIAL AND NOT COMPLETE.

Elementary logic addresses the ANDs, IFs, ORs and NOTs embedded in our spoken and written language. When formulated as an exhaustive tabulation, elementary logic is truth tables. When formalized in an inferential system, it is propositional calculus. When formalized in an equational system, it is Boolean algebra. And when formalized in a spatial system of containers, it is boundary logic.

Each of these systems of logic includes a set of rules or transformations which permit logical forms to change structure without changing meaning. Truth tables require brute-force evaluation of every possible set of variable bindings. The primary operator in an inferential system is implication, usually expressed as *modus ponens*. In equational systems, the familiar match-and-substitute permits equals to be exchanged for equals. In boundary logic, the primary operations are void-substitution (erasure or deletion) and transparency (virtual erasure).

LOGIC FORMALISM

Throughout the evolution of axiomatic logic, mathematicians have attempted to reduce and simplify the basis set of assumptions necessary to define the semantics of logic. Logicians cast this basis within a framework of inference; the central connective in a logical axiom is implication. In contrast, algebraists cast the basis within an equational framework; the central connective is equality. The difference between the two approaches is that logical inference is one-directional, that A implies B does not mean that B implies A. Logical implication is asymmetric. As a consequence, the processes of logical deduction and proof must accumulate inferences as a database of facts implied by a set of premises. In contrast, the mechanisms of algebra, which we know from seventh grade algebra of numbers, allow the current state of a computation to be carried along within an equation. Algebraic deduction is bidirectional, using substitution of equals for equals as a primary computational mechanism.

A third axiomatic approach is to use recursive function theory, to define processes in terms of a base case and an inductive, recursive case. Function theory assumes an equational context.

The degree of representational and computational power embodied in each of these axiomatic approaches varies, with implicational logic being the most clumsy for computation, and recursion being the most elegant.

Independent of the axiomatic approach, the representation of logical form also strongly effects the power and clarity of axioms. Lakatos, in Proofs and Refutations, shows that the evolution of a mathematical concept is dynamic, with emphasis falling at one time on the axioms themselves, and at another times on the definitions which identify the data structures that the axioms refer to.

SYMBOLIC NOTATIONS

Formal Logic

Several sets of notation for the operations of primary logic have been introduced by the various founders of symbolic logic. Generally each invented his own notation, Thus, negation may be

Language	*Peano*	*Hilbert*	*Variations*	*Boolean*	*Boundary*
NOT P	~P	$\overline{P}$	¬P -P	P'	(P)

We will select a set of symbols with convenient typography herein, since indeed, in contrast to diagrammatic notations, typographical symbols are intended to be typographically convenient.

Connective	*Symbol*
NOT	¬
AND	&
OR	v
IMPLIES	->
IF AND ONLY IF	=

Implicational Basis

TRUE	p -> p
FALSE	F
NOT	p -> F
AND	(p -> (q -> F)) -> F
OR	(p -> F) -> q
IMPLIES	p -> q
IF AND ONLY IF	((p -> q) -> ((q -> p) -> F)) -> F

Sheffer Stroke

Both NAND and NOR are single connective bases for primary logic, although technically they require one ground token, such as FALSE, in symbolic systems. The symbol for NAND is called stroke, |, and is named after Sheffer although it was first formalized by Peirce. The symbol for NOR is called dagger, †, also accredited to Sheffer.

<table>
<tr><th>Connective</th><th>Stroke</th><th>Dagger</th></tr>
<tr><td>TRUE</td><td>TRUE</td><td>FALSE†FALSE</td></tr>
<tr><td>FALSE</td><td>TRUE|TRUE</td><td>FALSE</td></tr>
<tr><td>NOT</td><td>p|p</td><td>p†p</td></tr>
<tr><td>AND</td><td>(p|q)|(p|q)</td><td>(p†p)†(q†q)</td></tr>
<tr><td>OR</td><td>(p|p)|(q|q)</td><td>(p†q)†(p†q)</td></tr>
<tr><td>IMPLIES</td><td>p|(q|q)</td><td>((p†p)†q)†((p†p)†q)</td></tr>
</table>

Combinators

Proposed first by Schonfinkel in 1924, and refined later by others, this system demonstrates that variables are not a necessary part of logic.

<<implications of no variables>>

Here is the SKI calculus, introduced by Curry, Feys, and Craig in 1958. It is the same as Schonfinkel's calculus.

Ix = x
Kxy = x
Sfgx = fx(gx)

Note that I is redundant, since

I = SKK

Polish Notation

Developed by Lukasiewicz, this system requires no brackets, Each capital letter operator takes an exact number of arguments which follow.

Connective	*Polish*	*Symbolic Polish*
NOT	Np	¬p
AND	Kpq	&pq
OR	Apq	vpq
IMPLIES	Cpq	->pq
IF AND ONLY IF	Epq	=pq

AXIOMATIC SYSTEMS

FREGE'S AXIOMS

Frege was the first to formalize logic. He presented his work in his own diagrammatic form, Frege diagrams. It is very difficult to find mention of his notation without consulting is original manuscripts in German. In the nearly universal process of transcribing Frege diagrams to symbolic logic, the power of Frege's diagrammatic thinking has been almost completely lost.

Frege provided six axioms for simple logic. They are not independent. The third "axiom" is a theorem of the first two. The fourth, fifth, and sixth can all be reduced to a single axiom:

(¬p->¬q)->(q->p)

The first three define the stroke of implication, the next three define negation. Frege's notation builds in a basis of {NOT, IMPLIES} since it has not other representational forms.

1) a -> (b -> a)
2) (c -> (b -> a)) -> ((c -> b) -> (c -> a))
3) (c -> (b -> a)) -> (b -> (c -> a))
4) (b -> a) -> (¬a -> ¬b)
5) ¬¬a -> a
6) a -> ¬¬a

LUKASIEWICZ' AXIOMS OF INFERENCE

Lukasiewicz produced perhaps the most elegant set of axioms of inference in symbolic notation.

1) (p->q)->((q->r)->(p->r))
2) p->(¬p->q)
3) (¬p->p)->p

1) is transitivity of implication
2) is *consequentia mirabilis* -- Stoics
3) is self-contradictory conjunction -- John Duns the Scot

NICOD'S AXIOM FOR BOOLEAN ALGEBRA

In 1917, Nicod proposed a single axiom for primary logic. It is always possible to cast any set of axioms into one single form, simply by combining them by conjunction. The difficulty is then to find patterns which match the usually complex result. That is, one always then decomposes the single axiom into a more convenient group of theorems.

Nicod's formalization, based on the Sheffer stroke (NAND):

[p|(q|r)]|([t|(t|t)]|{(s|q)|[(p|s)|(p|s)]})

[p|(q|r)]| {(s|q)|[(p|s)|(p|s)]})

or

[p|(q|r)]|{(t->t)|[(s|q)->(p|s)]}

where p->q = [(p|(q|q)]

The variety of delimiters above is solely to distinguish application nesting.

HUNTINGTON'S AXIOMS FOR BOOLEAN ALGEBRA

Huntington's algebraic system was the first to unite logic with group theory.

	+ (OR)	* (AND)
Commutativity	a+b = b+a	ab = ba
Identity	a+**0** = a	a**1** = a
Complement	a+a' = **1**	aa' = **0**
Distribution	a+bc = (a+b)*(a+c)	a*(b+c) = ab+ac

Below, the parens notation of boundary logic and the symbolic notation of conventional inferential logic are compared to Boolean Algebra.

Parens

Commutativity	implicit in unstructured space	
Identity	a = a	a = a
Complement	a (a) = ( )	((a) a) =
Distribution	a ((b)(c)) = ((a b)(a c))	
	((a)(b c)) = ((a)(b)) ((a)(c))	

Natural Logic

	+ (OR)	* (AND)
Commutativity	$(a \lor b) = (b \lor a)$	$(a \,\&\, b) = (b \,\&\, a)$
Identity	$(a \lor F) = a$	$(a \,\&\, T) = a$
Complement	$(a \lor \neg a) = T$	$(a \,\&\, \neg a) = F$
Distribution	$(a \lor (b \,\&\, c)) = ((a \lor b) \,\&\, (a \lor c))$	
	$(a \,\&\, (b \lor c)) = ((a \,\&\, b) \lor (a \,\&\, c))$	

Inferential Logic

Boolean algebra is quite awkward in pure inferential logic. Even adding NOT and maintaining the equality sign, the resulting expressions are unfamiliar. Without algebraic equality, each equation would convert into the form

$$\neg((X \rightarrow Y) \rightarrow \neg(Y \rightarrow X))$$

	+ (OR)	* (AND)
Commutativity	$(\neg a \rightarrow b) = (\neg b \rightarrow a)$	$\neg(a \rightarrow \neg b) = \neg(b \rightarrow \neg a)$
Identity	$(\neg a \rightarrow F) = a$	$\neg(a \rightarrow \neg T) = a$
Complement	$a \rightarrow a = T$	$\neg(a' \rightarrow a') = F$

Distribution

$$\neg a \rightarrow \neg(b \rightarrow \neg c) = \neg((\neg a \rightarrow b) \rightarrow \neg(\neg a \rightarrow c))$$

$$\neg(a \rightarrow \neg(\neg b \rightarrow c)) = (a \rightarrow \neg b) \rightarrow \neg(a \rightarrow \neg c)$$

SPENCER-BROWN'S ARITHMETIC AND ALGEBRAIC AXIOMS

In Laws of Form (1967), Spencer-Brown made the unique contribution of formalizing the arithmetic of logic.

```
     ( )( ) = ( )              CALLING

     (( ))  =                  CROSSING

    (a (a)) =                  POSITION

a ((b)(c)) = ((a b)(a c))      TRANSPOSITION
```

Position is AND-complement in Huntington's system, while transposition is distribution.

KAUFFMAN'S SINGLE AXIOM

```
((a b)(a (b))) = a
```

Kauffman elegantly shows the relationships between inference, logical tautology and logical arithmetic in the following two dimensional diagram, for which both rows and columns are combined in space to create new structures.

```
( a  b )   ( a (b))    =   (a)

((a) b )   ((a)(b))    =  ((a))

   =          =             =

(    b )   (   (b))    =   ( )
```

BRICKEN'S COMPUTATIONAL RULES

```
(A ( )) =                   VOID OCCLUSION

A {B A} = A {B}             PERVASION
```

The third equation that characterizes Boundary Logic is

((A)) = A INVOLUTION

which can also be derived as a theorem given the definition of equality.

X = Y =def= (X Y) ((X)(Y))

by substituting

X = ((A)) and Y = A

To express Pervasion in symbolic notations, we must use a shallow form of the rule, and apply it repeatedly, spreading the rule over both axioms and proof.

A (B A) = A (B) SHALLOW PERVASION

Computational Rules Expressed as a Recursive Function

Propositional logic, aka formal deduction, aka rationality, can be expressed as a single recursive function. The base case is Void Occlusion:

({A} ()) = <void> *Base case*

The inferential case is Deep Pervasion:

{A} {B {A}} = {A} {B} *Inductive case*

The base case uses parens, ( ), as its language to express logical forms, and establishes the class of void equivalent forms.

The inductive case uses braces called deeparens, { }, as schema that refer to the set of all possible parens forms. The curly braces are necessary in order to represent a variable as having both an outside and an inside.

All Boolean forms can be constructed using this equation constructively. Should one begin with <void>, the base case permits all False forms to be generated. Should one begin with a mark, ( ), a functional variety of the base case, Dominion, permits all True forms to be generated.

{A} ( ) = ( ) *Base case, Dominion*

Forms are semantically equal simply because they are constructed from the same base, ( ) or <void>.

DIAGRAMMATIC LOGIC

HISTORICAL EVOLUTION

In general, the original explorers of the concepts of logic formulated their understandings in spatial, rather than symbolic form. Symbolic logic is relatively new, introduced first by Boole in 1854. In the latter half of the 19th century, during the rigorous formalization of logic, Venn, Peirce and Frege all developed inherently spatial representations. Russell's symbolic notation was adopted universally after around 1910, since the spatial forms were considered too clumsy to work with.

Aristotle's Square of Opposition

Aristotle wanted to create a classification of declarative language, presumably to enhance the veracity of Greek debate.

The Square of Opposition was the first spatial display of logical concepts:

	Affirm	*Deny*
Universal	**A** Every _ is _.	**E** No _ is _.
Particular	**I** Some _ is _.	**O** Some _ is not _.

In terms of parens:

	Affirm	*Deny*
Universal	(_) _	((_) _)
Particular	((_)(_))	(_)(_)

Syllogistic Figures

B is the minor term
M is the middle term
C is the major term

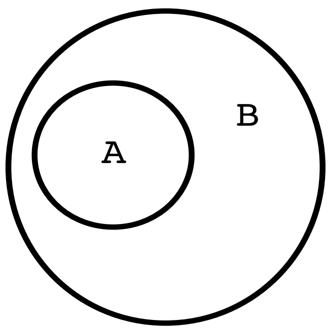
_ is one of the AEIO forms from the Square of Opposition. B, M and C fill in the blanks in the particular AEIO form.

Figure	*First*	*Second*	*Third*	*Fourth*
Major premise	M_C	C_M	M_C	C_M
Minor premise	B_M	B_M	M_B	M_B
Conclusion	B_C	B_C	B_C	B_C

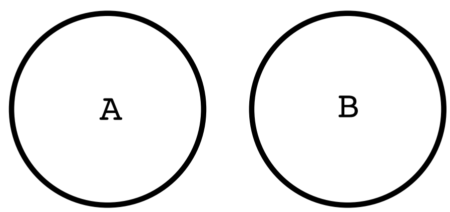
The moods of the figure are the particular choice of the AEIO form in the figure's "shape".

Euler Diagrams

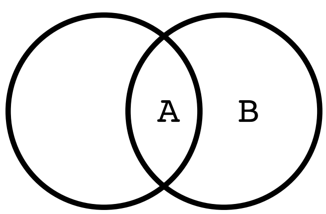
The first use of closed planar circles to represent syllogistic relations probably occurred in the 16th century [Lull, Ars Magna]. In the eighteenth century, Euler proposed a system which used spatial enclosure to express the syllogistic figures. At that time, logic was solely syllogistic reasoning. There is no way to express XOR in this language.



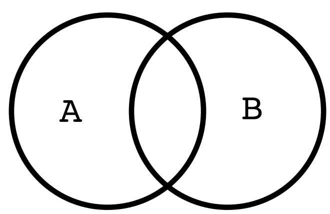
All A is B
(A) B



No A is B
((A) B)



Some A is B.
((A)(B))

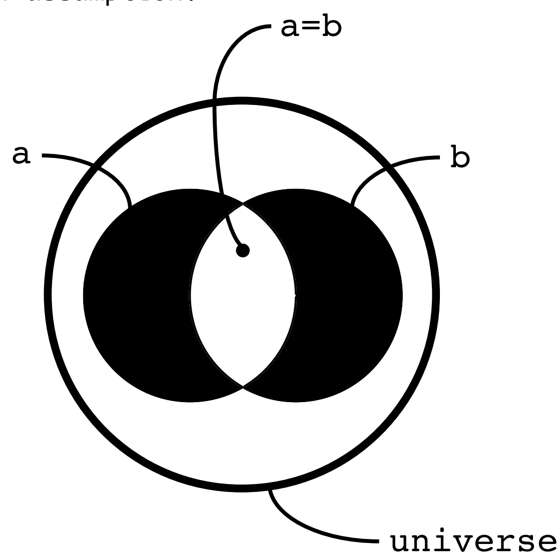


Some A is not B.
(A)(B)

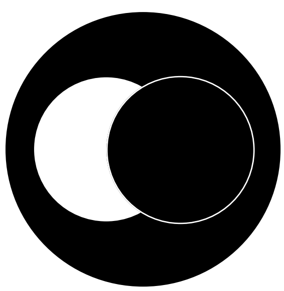
Euler was limited by following Aristotle too closely.

Venn Diagrams

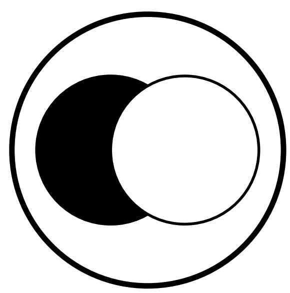
The structural variation does show up as a cloven black-space. The two pieces of black are mandatory so that we can tell the difference between a and b. Appearance of structural variation in XOR since rotation of the figure is a hidden assumption.



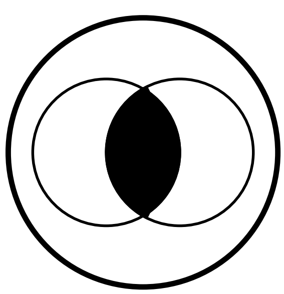
Here is a comparison of Venn's fully expressible system, compared to Euler's limited system with no composability operators.



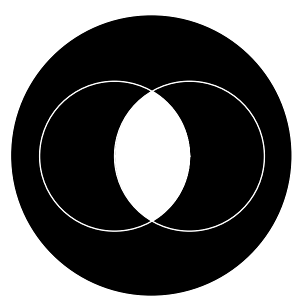
All A is B
(A) B



No A is B
((A) B)



Some A is B.
((A)(B))



Some A is not B.
(A)(B)

Truth Tables

XOR		b: F	b: T
a	F		T
	T	T	

Truth tables are a tabular layout for exhaustively determining the value of a particular propositional expression.

We will illustrate the cases of truth for the XOR function expressed as a composition of other simple logical connectives.

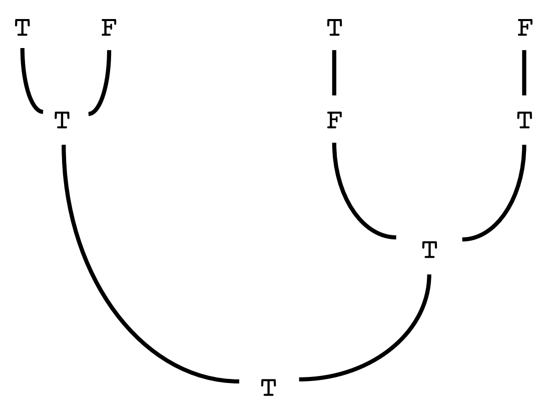
(a XOR b) = ((a OR b) AND ((NOT a) OR (NOT b)))

a	*b*	*not a*	*not b*	*a OR b*	*-a OR -b*	*a XOR b*
0	0	1	1	0	1	0
0	1	1	0	1	1	1
1	0	0	1	1	1	1
1	1	0	0	1	0	0

Evaluation Trees

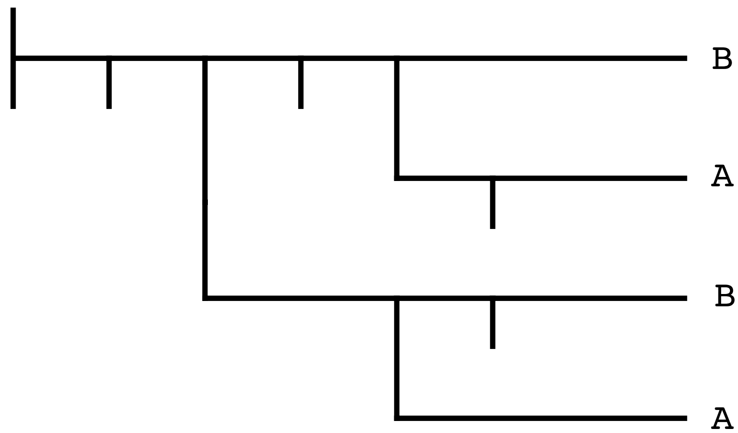
The evaluation of a propositional expression, given the values of the input variables, takes the form of a tree:

((a OR b) AND ((NOT a) OR (NOT b)))

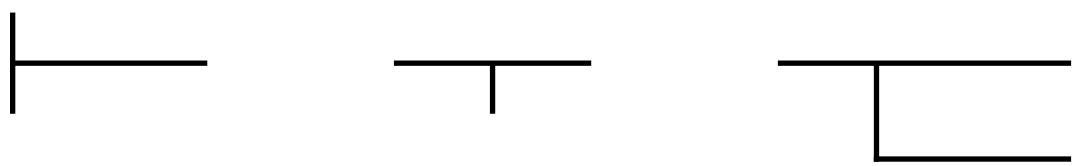


Frege Diagrams

Frege originated much of our modern approach to formal logic. He invented a spatial notation to express "formal thought".



Frege diagrams are read right to left, with the inputs on the far right and the output on the far left. The diagram is a single form composed three elements.



Evaluation = ( ) Negation ( ) Implication (_) _

Evaluation is the termination condition for the diagram, the output. The vertical stroke is the actual end-point, the horizontal stroke is the content stroke, the form of the expression itself.

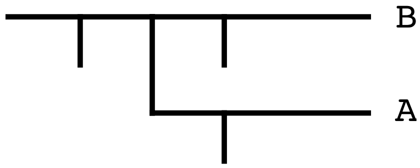
A vertical negation bar under the content stroke inverts the value on the right as it "flows" to the left.

A vertical bar connected below to another horizontal content bar links the two content strokes by implication. The lower content implies the upper content.

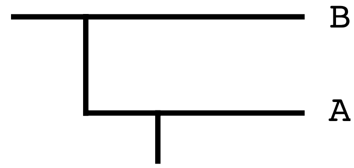
Composition of logical expressions is by attaching the left-side of one content stroke to the right-side of another.

The language elements have subtle interconnections. For instance, negation is the same as implication with an empty lower content stroke. Negation can be read as an implication for which the content is the upper stroke rather than a new lower stroke.

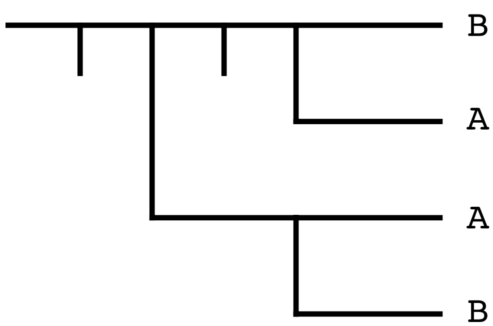
Some common logical connectives follow:



A AND B



A OR B



A XOR B

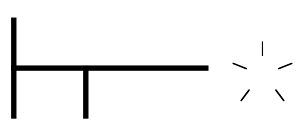
The generic structure of IMPLIES has three locations for a negation bar

on the far left
on the top stroke
on the bottom stroke

This defines 8 of the 16 Boolean connectives. The four single value connectives are single strokes ending on the right with either a or b, and either having or not having a negation stroke. What remains is the two compound connectives XOR and IFF (shown above), and the two truth values.

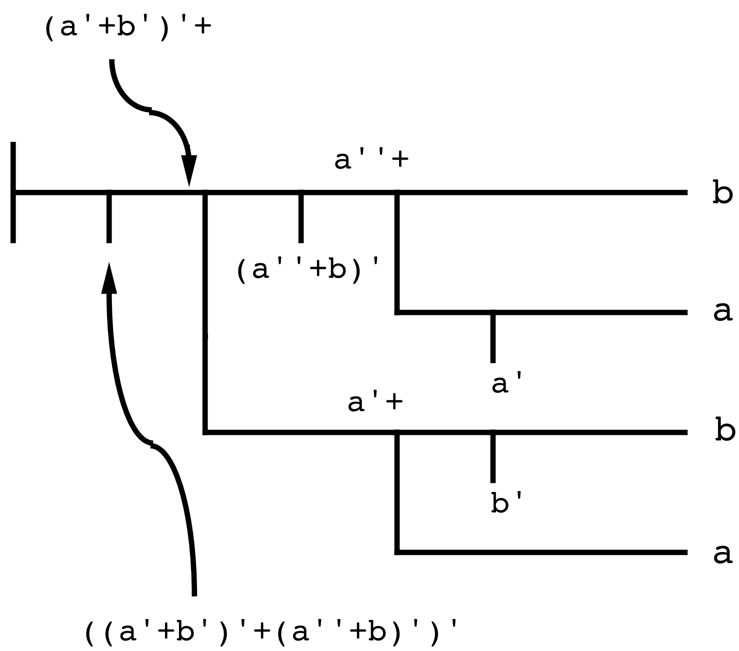
How are TRUE and FALSE indicated? By single strokes without a terminal label. The evaluation stroke conveys TRUE and a negation bar on the unlabelled stroke is FALSE.

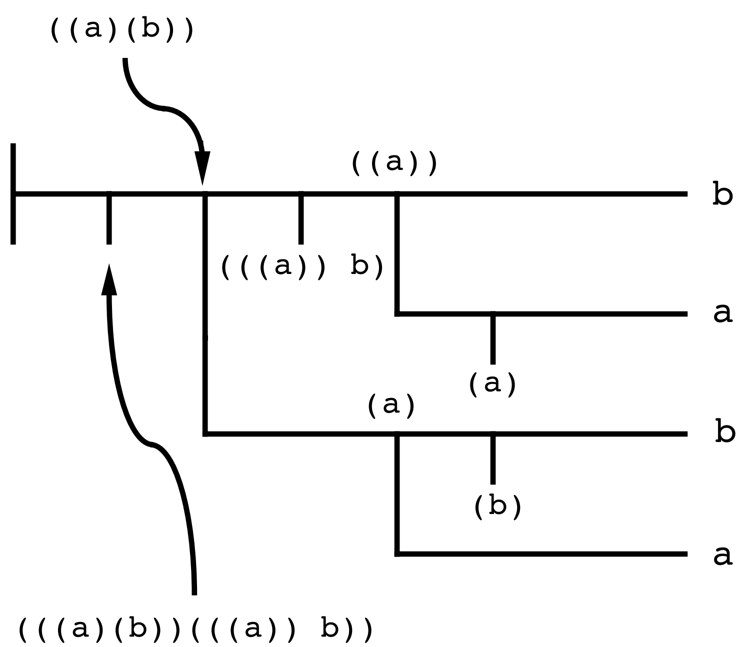
TRUE



FALSE

This diagrammatic formal language was never adopted by the logic community, although it was invented by the same person who invented formal logic. The diagrammatic notation maps directly onto parens,





REPRESENTATIONAL DIVERSITY

DIAGRAMMATIC FORMS IN BOUNDARY LOGIC

- Enclosures
  - Parens
  - Capped Parens
  - Boxes
  - Circles
- Graphs
  - Parens with Depth
  - Parens Extruded Downward
  - Parens Trees
  - Distinction Trees
  - Distinction Networks
  - Cyclic Distinction Networks
  - Crossbound Graphs
- Maps
  - Distinction Steps
  - Blobby Distinction Maps
  - Circular Distinction Maps
  - Rectangular Distinction Maps
  - Distinction Rooms
- Perspective
  - Centered Distinction Steps
  - Centered Circular Distinction Maps
  - Centered Rectangular Distinction Maps
- Blocks
  - Parens Extruded Upward
  - Distinction Stacks
  - Distinction Walls
  - Distinction Blocks
- Paths
  - Bar Trees
  - Bar Graphs
  - Path Graphs
  - Distinction Paths

BOUNDARY LOGIC

ENCLOSURES

Parens

((a b)((a)(b)))

Parens notation uses presence and absence to indicate TRUE or FALSE. There are 8 ways to parenthesize two labels in pairs, four ways to parenthesize two single labels, and two ways with no labels, one of which is the absence of any mark.

	splat	( )	
a	(a)	b	(b)
a b	(a) b	(b) a	(a)(b)
(a b)	((a) b)	((b) a)	((a)(b))

What remains are the two complex forms XOR and IFF, one of which is simply the bounded version of the other.

How can we select a pair of simple forms above so that they do not cancel parts of each other out? Given the laws of Boundary Algebra (later), only two pairings are possible:

The pairs which do not cancel must necessarily be bounded with two variables. This means they must be selected from the row:

(a b) ((a) b) ((b) a) ((a)(b))

Further the bounding of each variable must not be duplicated. This leaves

(a b) ((a)(b))

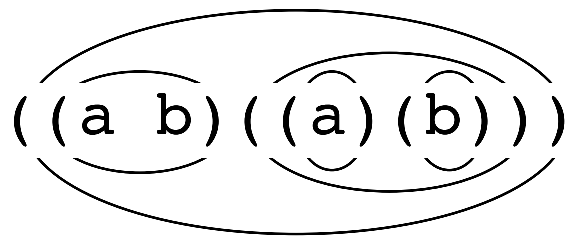
and

((a) b) ((b) a)

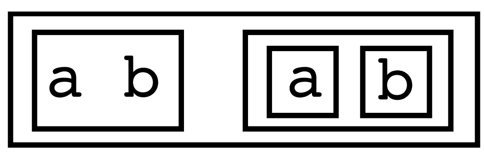
which are the two structural varieties of XOR/IF.

Capped Parens

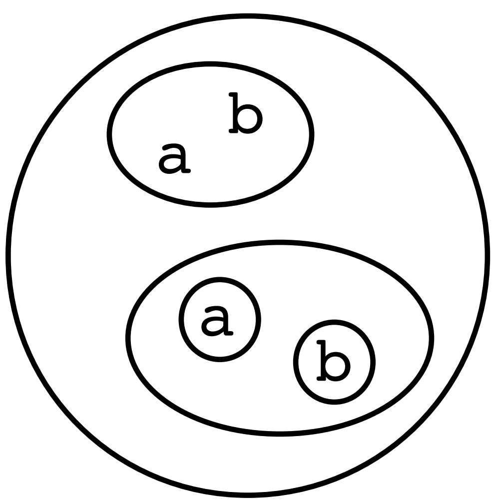
Enclosing a parens form is simply connecting the top and bottom of each parens pair to create an two-dimensional circular or rectangular enclosure. Parens structures convert to nested, non-intersecting enclosures.



Boxes



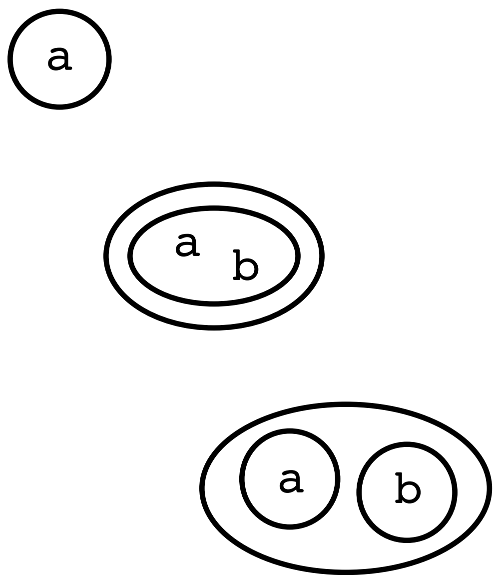
Circles



Extruding a parens form is extending each depth into a second dimension, to create a graph or a stack of boundaries.

These forms assume a viewing perspective "from above". They do not support commutativity

The main connectives (NOT, OR, AND):

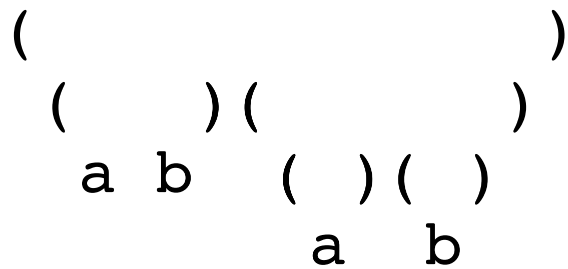


GRAPHS

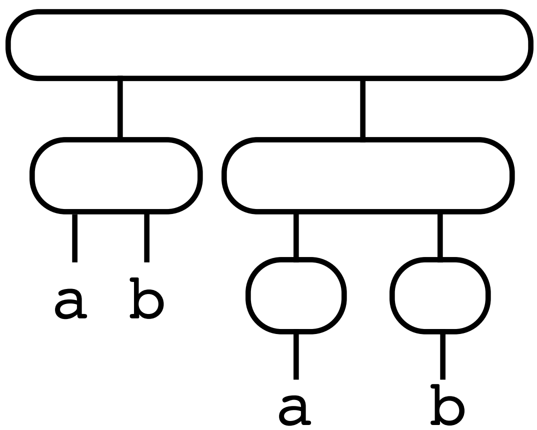
Parens with Depth

((a b) ((a) (b)))

Parens Extruded Downward

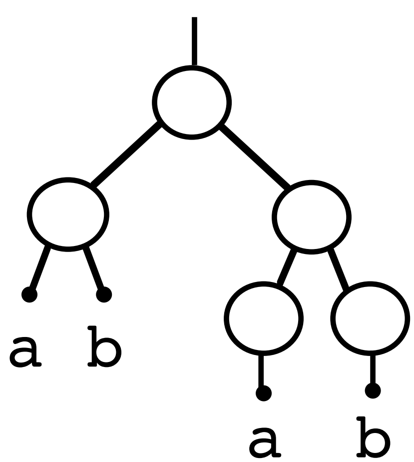


Parens Trees

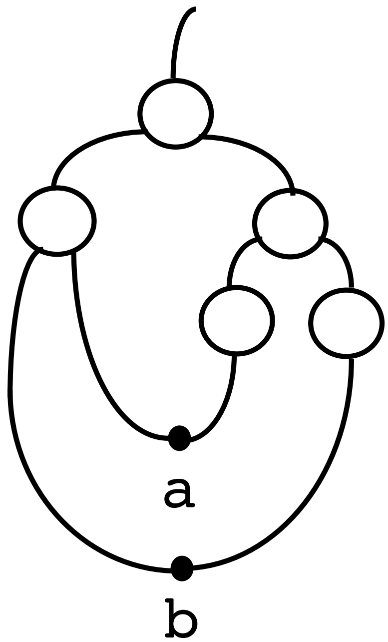


Distinction Trees

The apparent commutativity is an illusion, since inputs are not ordered.



Distinction Networks

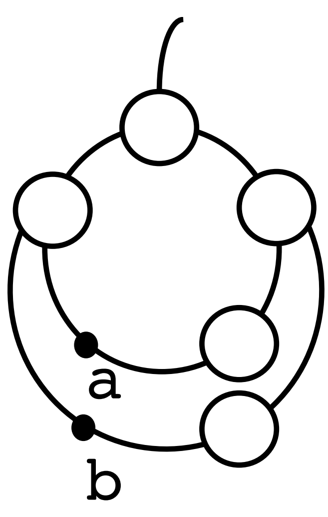


We have shifted the viewing perspective to assume top-to-bottom (or bottom-to-top if you prefer). That is the meaning of the little stem on the graphs.

The difference between a textual parens form and a graph is simply a 90 degree rotation of perspective out of the page.

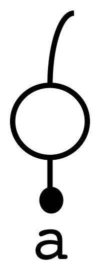
This notation presents the first occurrence of a planarity question, since connective arcs can potentially cross.

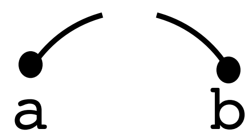
Cyclic Distinction Networks

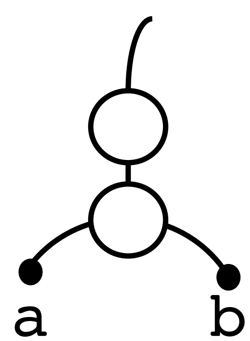


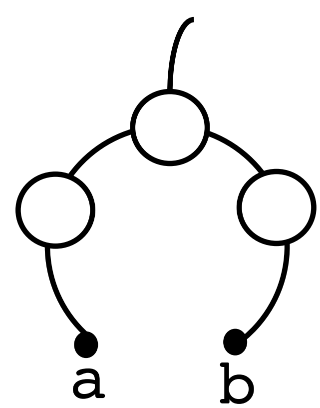
Flow is implicitly down in cycles, with overlaying of curves setting ordering.

The main connectives (NOT, OR, NOT-NOT, AND):



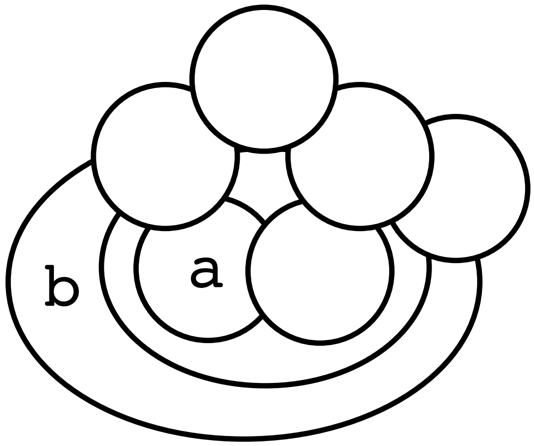


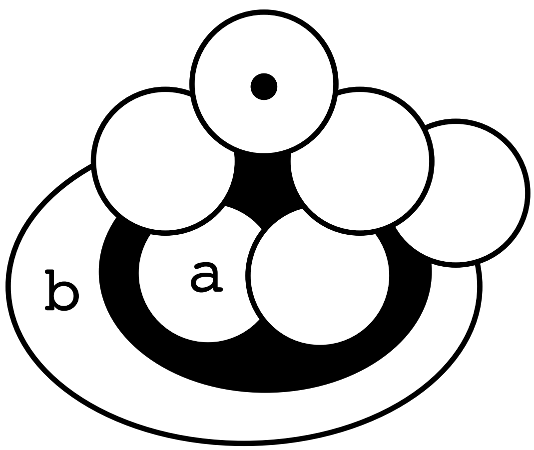




MAPS

Distinction Steps

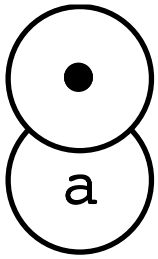


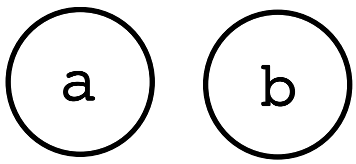


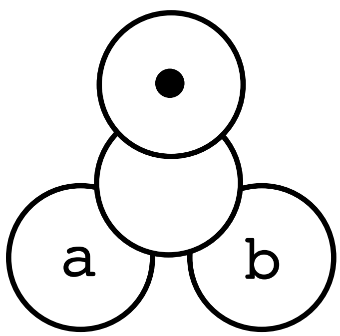
Graphs are viewed from outside, using a external to the representation point-of-view. Maps can be viewed from the inside, since the location of the point-of-view determines the meaning of the logic map.

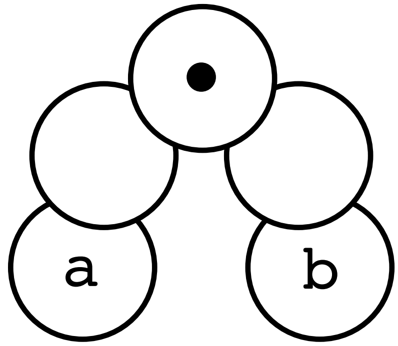
They do not support commutativity.

The main connectives (NOT, OR, NOT-NOT, AND):

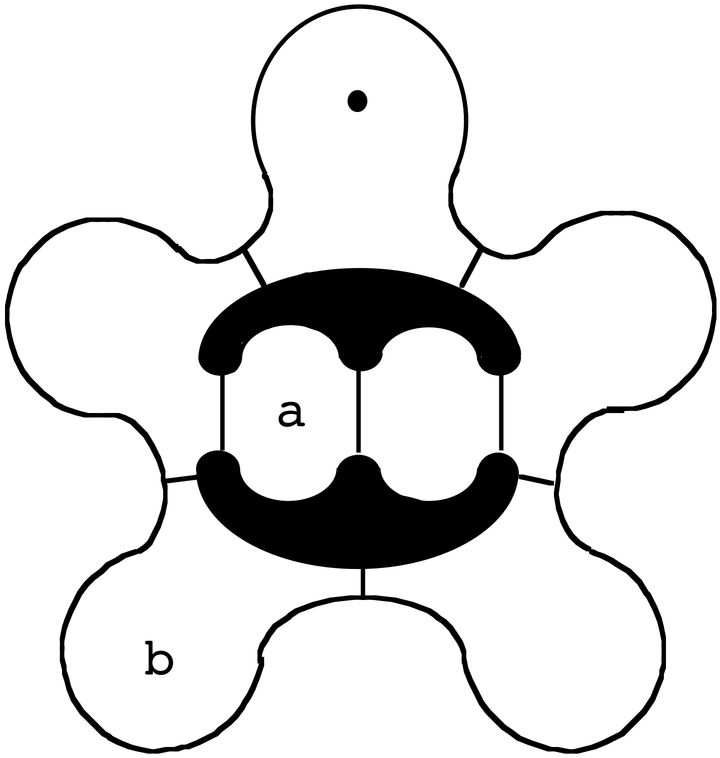




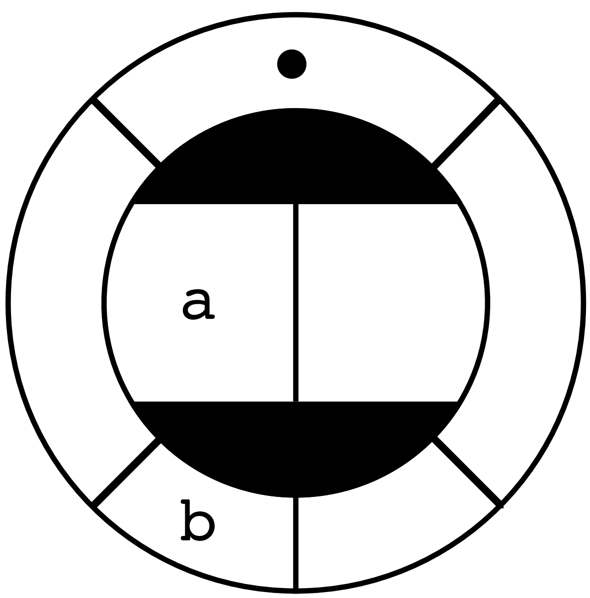




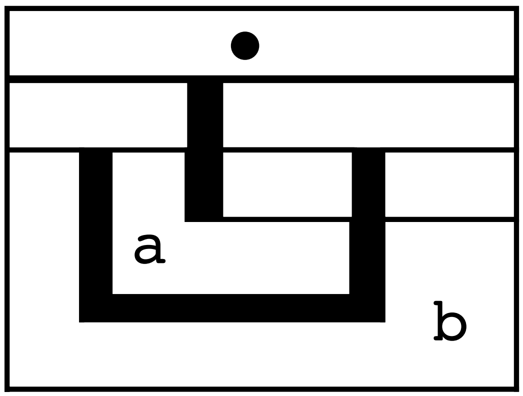
Blobby Distinction Maps



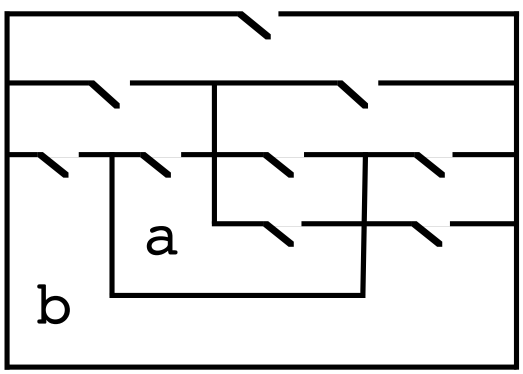
Circular Distinction Maps



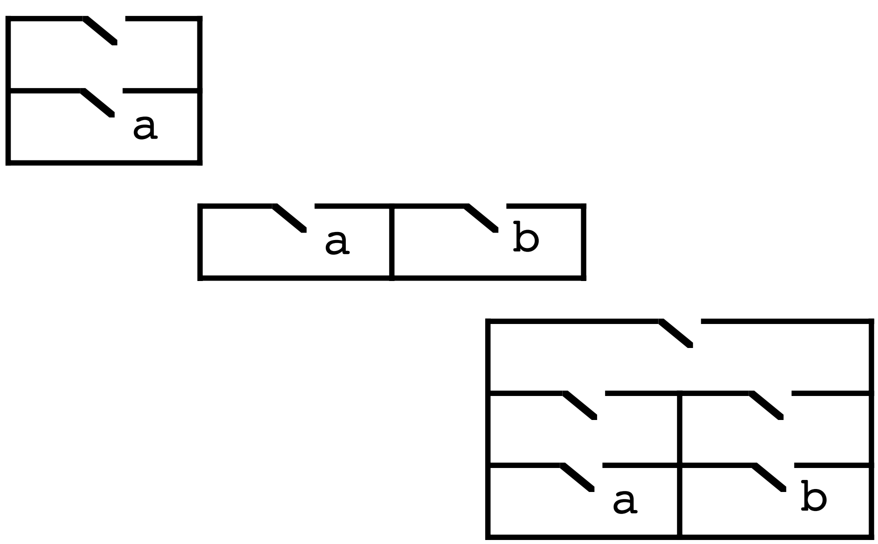
Rectangular Distinction Maps



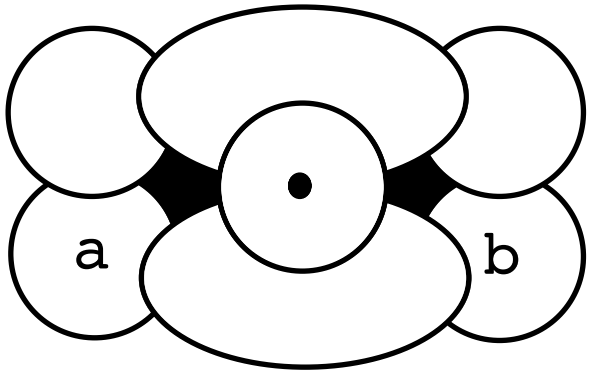
Distinction Rooms



The main connectives (NOT, OR, AND):



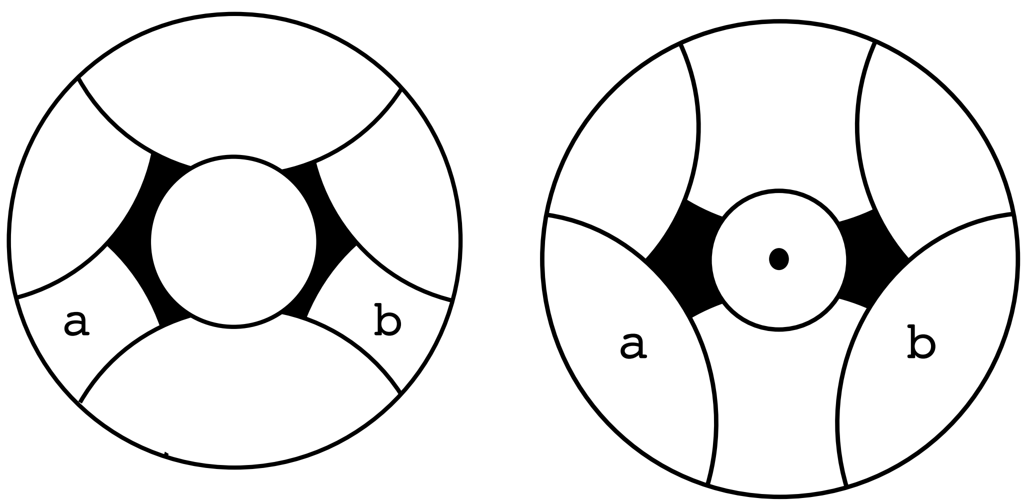
Centered Distinction Steps



The perspective-location dot is not strictly necessary, however in the following, we abandon depth cues.

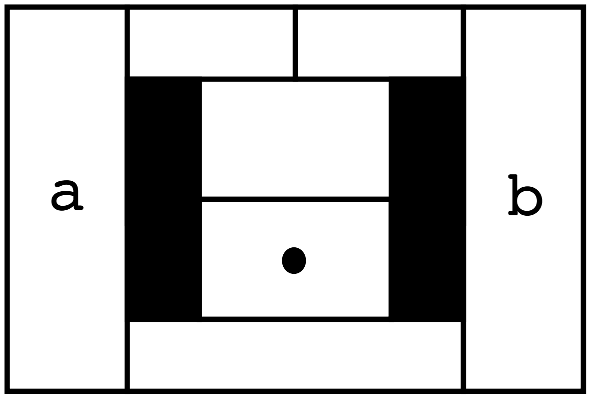
Centered Circular Distinction Maps

This notation moves from conventional perspective to none,



Centered Rectangular Distinction Maps

Giving up the visual overlap cue of circular forms:

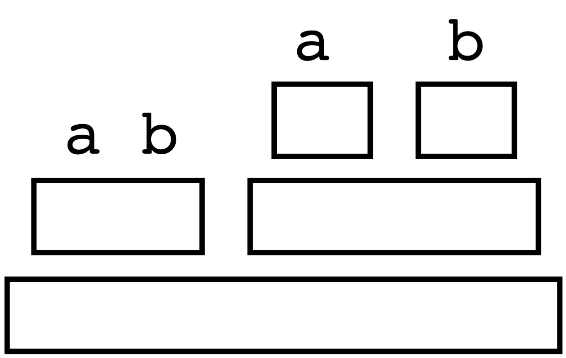


BLOCKS

Parens Extruded Upward

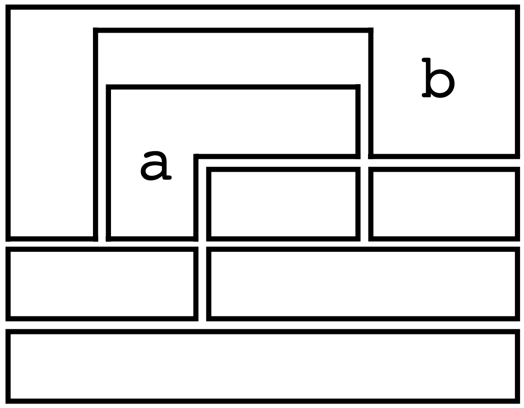
```
             a   b
   a b      ( )( )
  (    )(         )
 (                  )
```

Stacked Distinctions



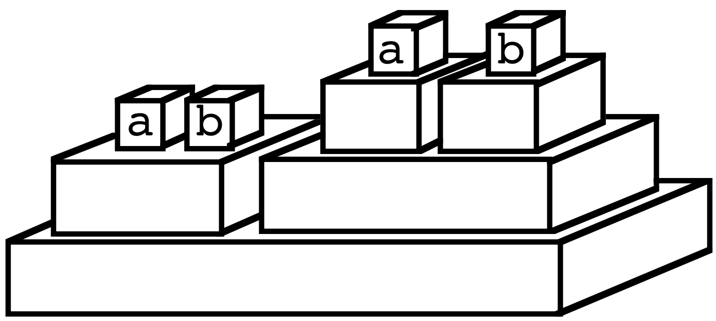
Assumed top-to-bottom.

Distinction Walls

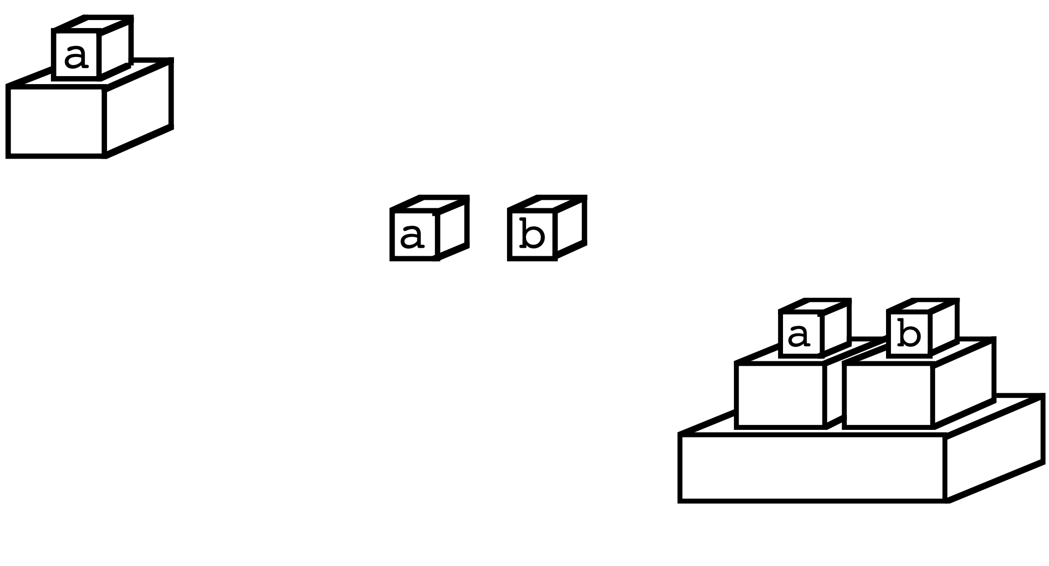


Single objects begin to remove the top-to-bottom assumption.

Distinction Blocks

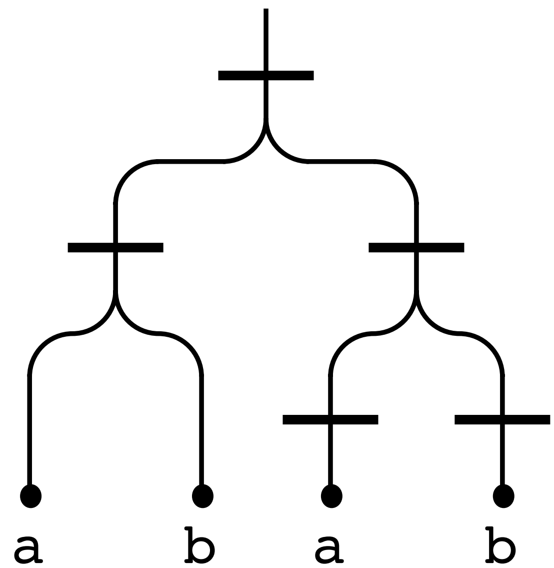


The main connectives (NOT, OR, AND):



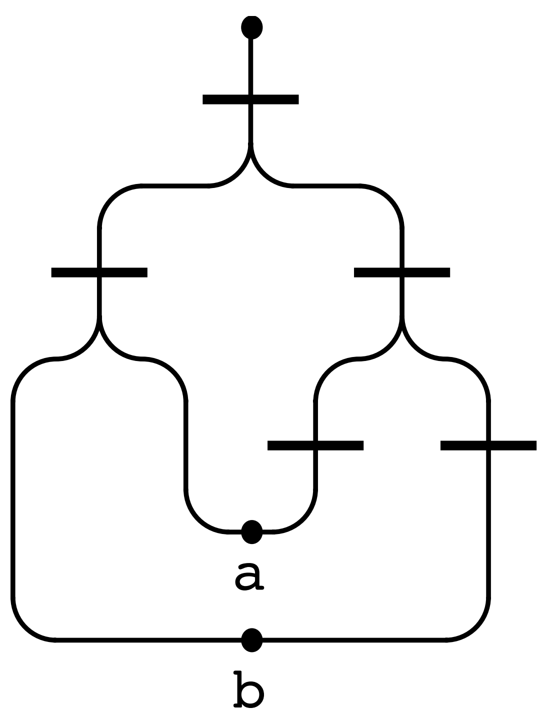
PATHS

Bar Trees

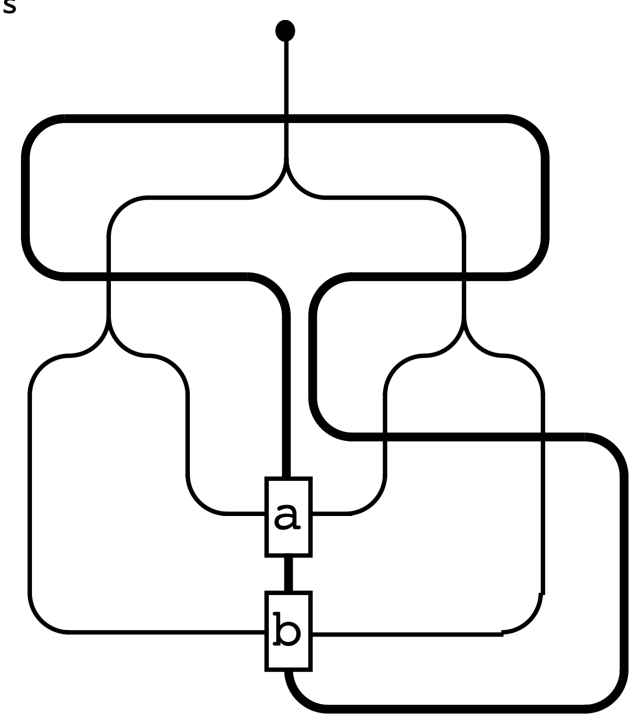


Bar Graphs

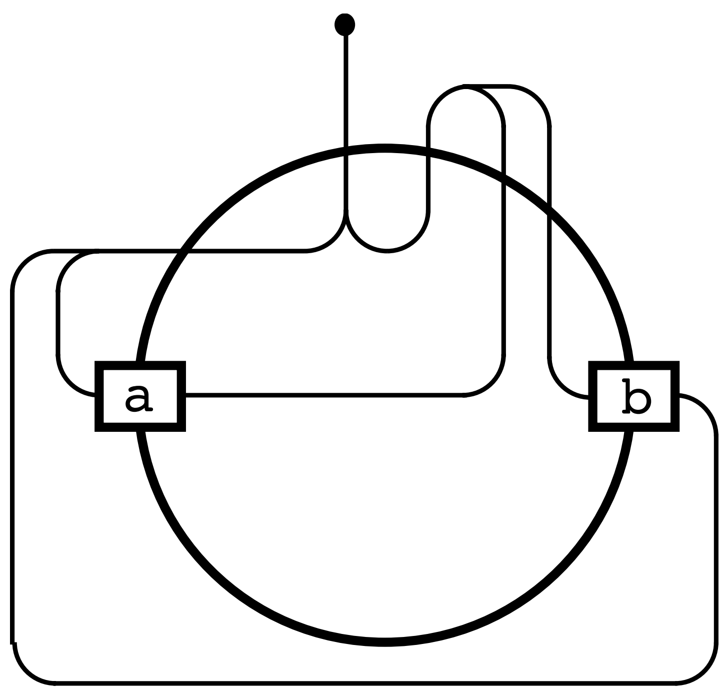
Do not support commutativity



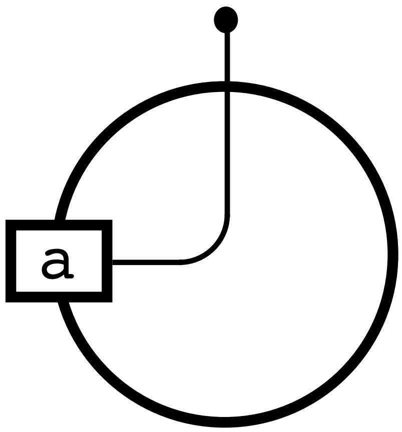
Path Graphs

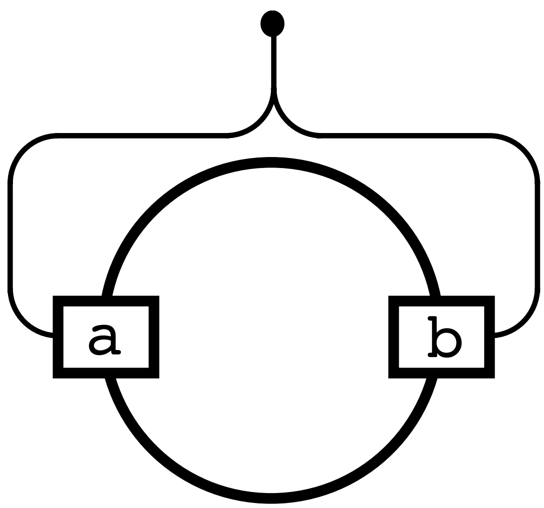


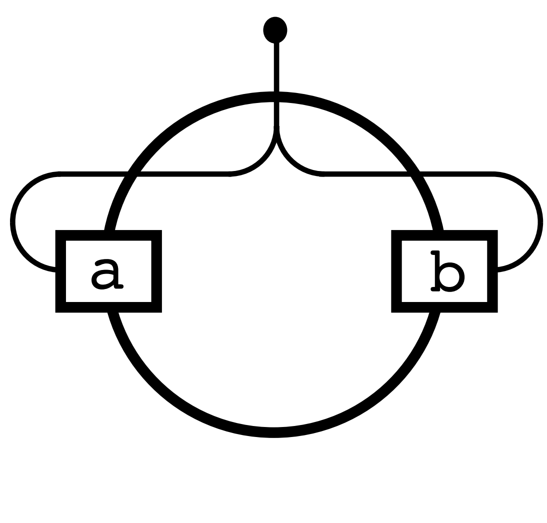
Distinction Paths



The main connectives (NOT, OR, AND):







NONSYMBOLIC LOGIC TRANSFORMATION TOOLS

- Modern Tools
  - Natural Deduction
  - Resolution
  - Logic State Machines
  - Boolean Cubes
  - Matrices
  - Lattices
- Computers
  - Logic Networks
  - Switching Circuits
  - Circuit Schematics
  - Transistor Networks

MODERN TOOLS

Axiomatic Systems

1847 DeMorgan
1854 Boole
1879 Frege
c1880 Peirce
1881 Venn
1894 Peano
1904 Huntington
1910 Russell
1913 Sheffer
1917 Nicod
1921 Wittgenstein
1924 Schonfinkel
1927 vonNeumann
1930 Lukasiewicz
1933 Huntington
1934 Hilbert and Bernays
1934 Carnap
1934 Gentzen

Natural Deduction

In the following spatial notation, [P] means "suppose that P is TRUE"

The bottom of each notational grouping supports the Top, traveling down is deduction.

Natural deduction has balanced rules for introducing and eliminating signs.

```
P   Q           P&Q             P&Q
-----           ---             ---
 P&Q             P               Q


 P             Q
---           ---
PvQ           PvQ


      [P]  [Q]
PvQ    Q    R
-------------
    R


 [P]
FALSE          P    ¬P             ¬¬P            FALSE
-----          -------             ---            -----
 ¬P             FALSE               P               P


[P]
 Q            P    P->Q
----            ---------
P->Q                Q
```

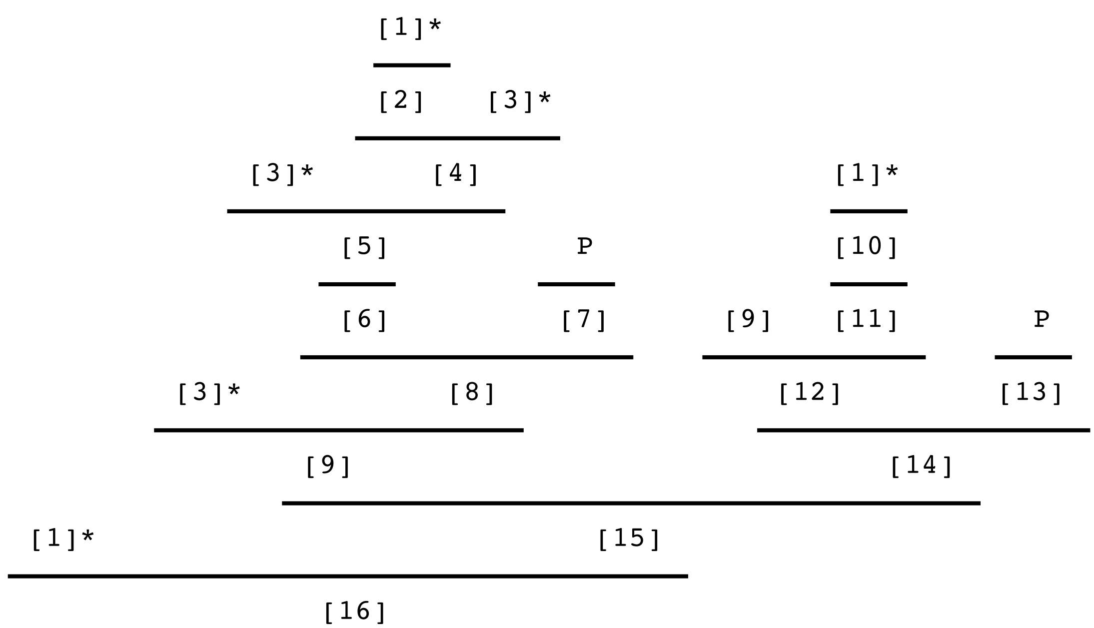
Proof Steps in Natural Deduction

Logical proof using natural deduction consists of creating a list of justified steps. Since the results from any prior step may be called upon in a following step, the steps themselves are not a linear ordering, but rather they are a spatial array.

```
Premise:      ((a OR b) AND ((NOT a) OR (NOT b)))
Conclusion:   (NOT ((IF a b) AND (IF b a)))
```

```
------
1.  ((IF a b) AND (IF b a)))             indirect proof, assumption
2.  (IF a b)                             simplification 1
---
      3.  a                                    indirect proof, assumption
      4.  b                                    modus ponens 2,3
      5.  (a AND b)                            conjunction 3,4
      6.  (NOT ((NOT a) OR (NOT b)))           DeMorgan 5
      7.  ((NOT a) OR (NOT b))                 simplification, premise
      8.  (NOT ((NOT a) OR (NOT b)))
            AND ((NOT a) OR (NOT b))           conjunction 6,7
      9. (NOT a)                               contradiction 3,8
---
10. (IF b a)                             simplification 1
11. (IF (NOT a) (NOT b))                 contraposition 10
12. (NOT b)                              modus ponens 9,11
13. (a OR b)                             simplification, premise
14. a                                    disjunctive syllogism 12,13
15. a AND (NOT a)                        conjunction 9,14
16. (NOT ((IF a b) AND (IF b a)))        contradiction 1,15  QED.
------
```

In recognition of the non-linear structure of proofs, logicians use a tree format for display which highlights the partial ordering between premises and conclusion. Conversion of sequential display into tree form is called "resolution into proof threads". The above proof



Resolution

In 1965, John Robinson devised a computational approach to proof called ***resolution***. The resolution principle uses the two possible cases of one variable. If the fact that a thing is True leads to one conclusion, and the fact that it is False leads to another conclusion, then in any case either the first or the second conclusion is True.

((if P then Q) and (if (not P) then R)) implies (Q or R)

As a deductive rule, resolution can be stated in increasing general forms:

P	and	¬P or False	|=	False
P	and	¬P or Q	|=	Q
P or Q	and	¬P or Q	|=	Q
P or Q	and	¬P or R	|=	Q or R
(P and U) or Q	and..	(¬P and V) or R	|=	(U or Q) or (V or R)

Resolution proof uses a ***clausal*** data structure consisting of sets of literals in disjunction. A pair of sets, one with a positive occurrence of a variable and one with a negative occurrence, is resolved by forming the union of the two sets, and deleting the resolvent variable.

{p, q} union {¬p, r} |= {q, r}

Facts are expressed as a singular set:

{p}

Rules are converted from implicational form to disjunctive form:

p -> q ==> ¬p or q ==> {¬p, q}

When the resolvent atoms have internal structure (functions and relations), the internal variables are ***unified*** in the course of resolving the atoms.

Resolution expresses Boolean functions as sets of literals. This is a different way to express CNF. The disjunctive forms in each clause form a set with implicit disjunction. Each clause forms a different set.

Literals: atoms and negated atoms

Clauses: sets of literals joined by OR

The Resolution Rule

Let S1 and S2 be sets of clauses, and U be the set Union operator:

({a,b,...} U S1) & ({¬a,b,...} U S2) ==> {b,...} U S1 U S2

E.g.: {x,b,¬c} & {¬x,b,d} ==> {b,¬c,d} resolve on x.

Termination

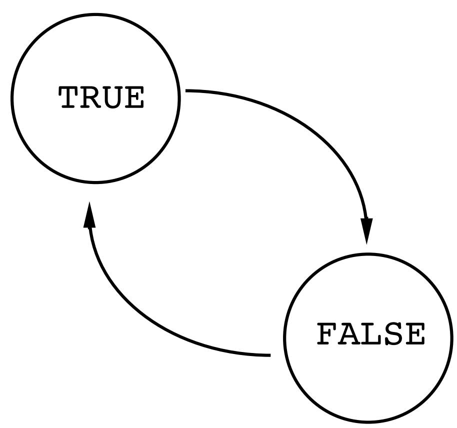
{a} & {¬a} ==> { } ==> False

{a,¬a} ==> True

Not complete

¬{¬a,¬b} & { } ==> no action

Logic State Machines



Two objects, 16 ways to connect with presence or absence of directional arrows.

Boolean Cubes

A Boolean function can be expressed in terms of a collection of vertices of a hypercube (*this is not the same use as the lattice hypercube*). The set of all Boolean functions of N variables is defined by all the possible collections (the power set) of vertices (called **cubes**).

Each cube is the *conjunction of unique literals*, one from each variable. The whole is formed by the disjunction of all cubes.

Examples:

```
                       a   ¬a
1 variable             •---•


                      a&b    a&¬b
2 variables                     •---•
                       |    |
                       •---•
                     ¬a&b    ¬a&¬b
```

Boolean Cube Operations

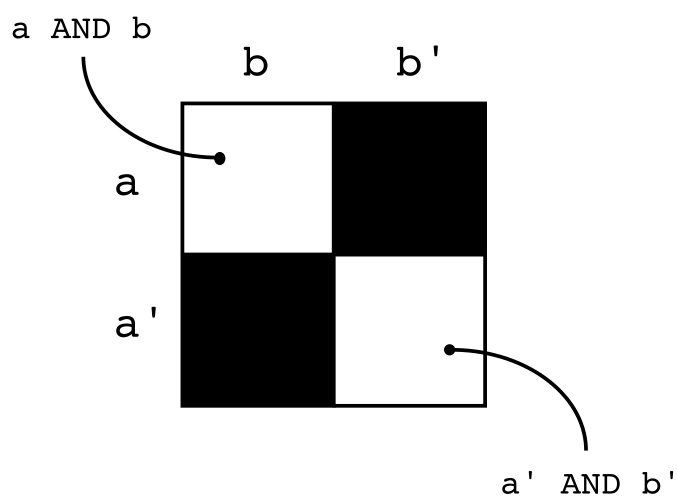
Cubes can be used for computation, either symbolically or physically.

function = set of cubes

not function = set of cubes not in function

f or g = overlay the cubes of f and the cubes of g

f and g = intersect the cubes of f and the cubes of g



Matrix Logic

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

By arranging the truth table of a Boolean function in a matrix form, the rules of logic can be converted into the rules of matrix algebra. The general format is:

		B	
		T	F
	T	•	•
A			
	F	•	•

Some examples:

A & B	A v B	A = B	A->B	A	¬A	T
1 0	1 1	1 0	1 0	1 1	0 0	1 1
0 0	1 0	0 1	1 1	0 0	1 1	1 1

Each Boolean matrix is an ***operator***.   That is, in this formulation, there are no objects. When using binary operations, matrix addition is xor; matrix multiplication is and.

a + b = c	a * b = c
0 + 0 = 0	0 * 0 = 0
0 + 1 = 1	0 * 1 = 0
1 + 0 = 1	1 * 0 = 0
1 + 1 = 0	1 * 1 = 1
xor	and

Note that these relations are the same ones that apply to computational addition.

As well, some matrix combinations result in matrices which are not Boolean functions. This then extends Boolean operations into generally unexplored territory, *imaginary Boolean operations*. Some examples of translating between operators:

```
a + ¬a = T                    1 1  +  0 0  =  1 1
                              0 0     1 1     1 1

a * ¬a = a                    1 1  *  0 0  =  1 1
                              0 0     1 1     0 0

xor + and = or                0 1  +  1 0  =  1 1
                              1 0     0 0     1 0

nor^2 = nor                   0 0  *  0 0  =  0 0
                              0 1     0 1     0 1

xor^2 = equal                 0 1  *  0 1  =  1 0
  (square-root of equal)      1 0     1 0     0 1

and + or = ?                  1 0  +  1 1  =  2 1
                              0 0     1 0     1 0
```

Lattices

A lattice is a *directed graph* with links representing an ordering relation. Lattices can have a maximal and a minimal element

```
( )   maximal
 |
( )
 |                 > greater than
( )
 |
( )   minimal
```

A ***partial ordering*** uses the ordering relation *greater-than-or-equal-to.*

```
   ( )        maximal
  /   \
(1)   (2)                 1 = 2
  \   /
   ( )        minimal
```

Hasse Diagrams (aka lattices)

A *set* and an *ordering relation* {S,>}, such that

- each object is a *vertex*
- if (a > b), then a is *higher than* b.
- if there is no c such that (a > c > b), then a *is connected to* b.

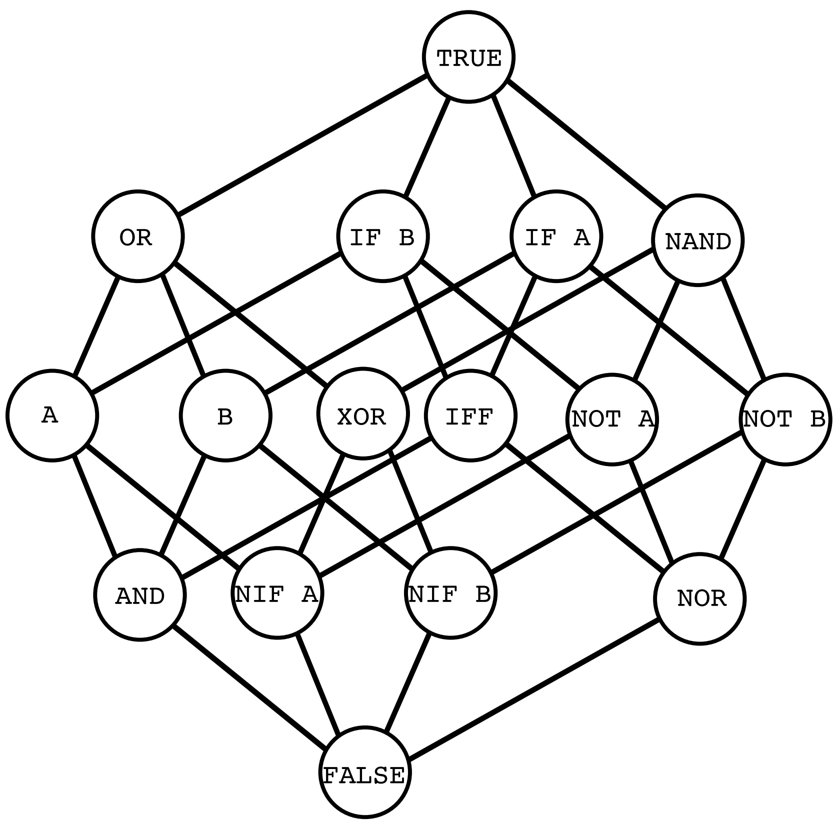
```
(a)    (b)     no maximal element
  \    /
   (c)         (a>c) & (b>c) & (c>d)
    |
   (d)         minimal
```

Boolean Hypercube

A distributed, complemented lattice. Two lattice combinational operations, meet (follow any two lines down) and join (follow any two up).

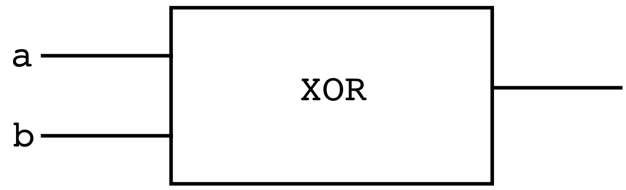
For a Boolean interpretation, join = AND, meet = OR

Tells all structural relations between different binary logic operators.

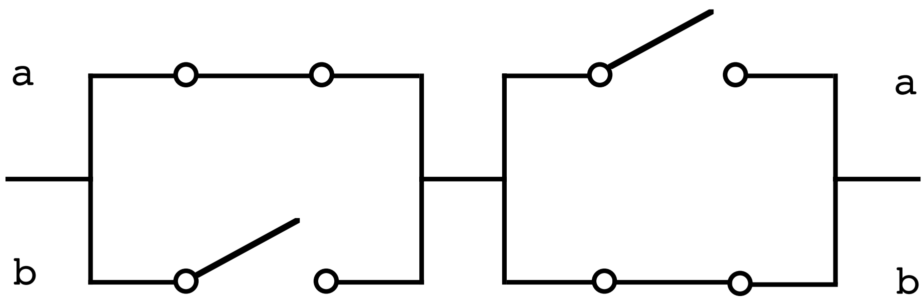


COMPUTERS

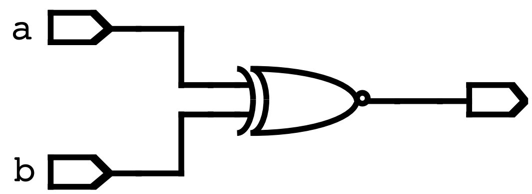
Logic Networks

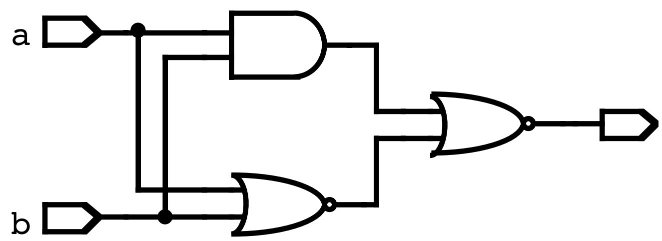


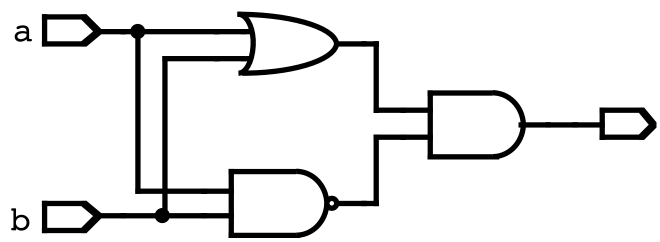
Switching Circuits

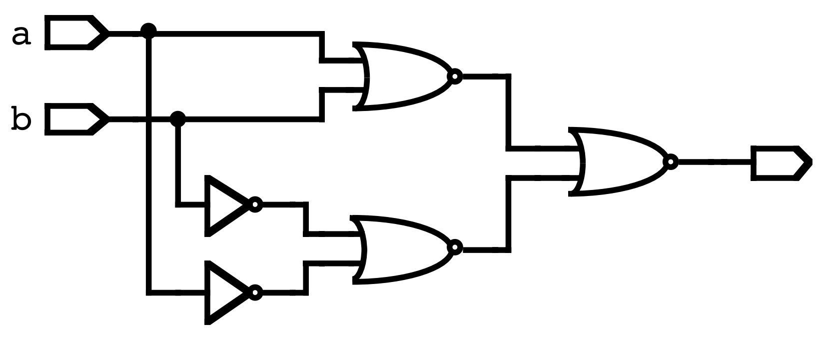


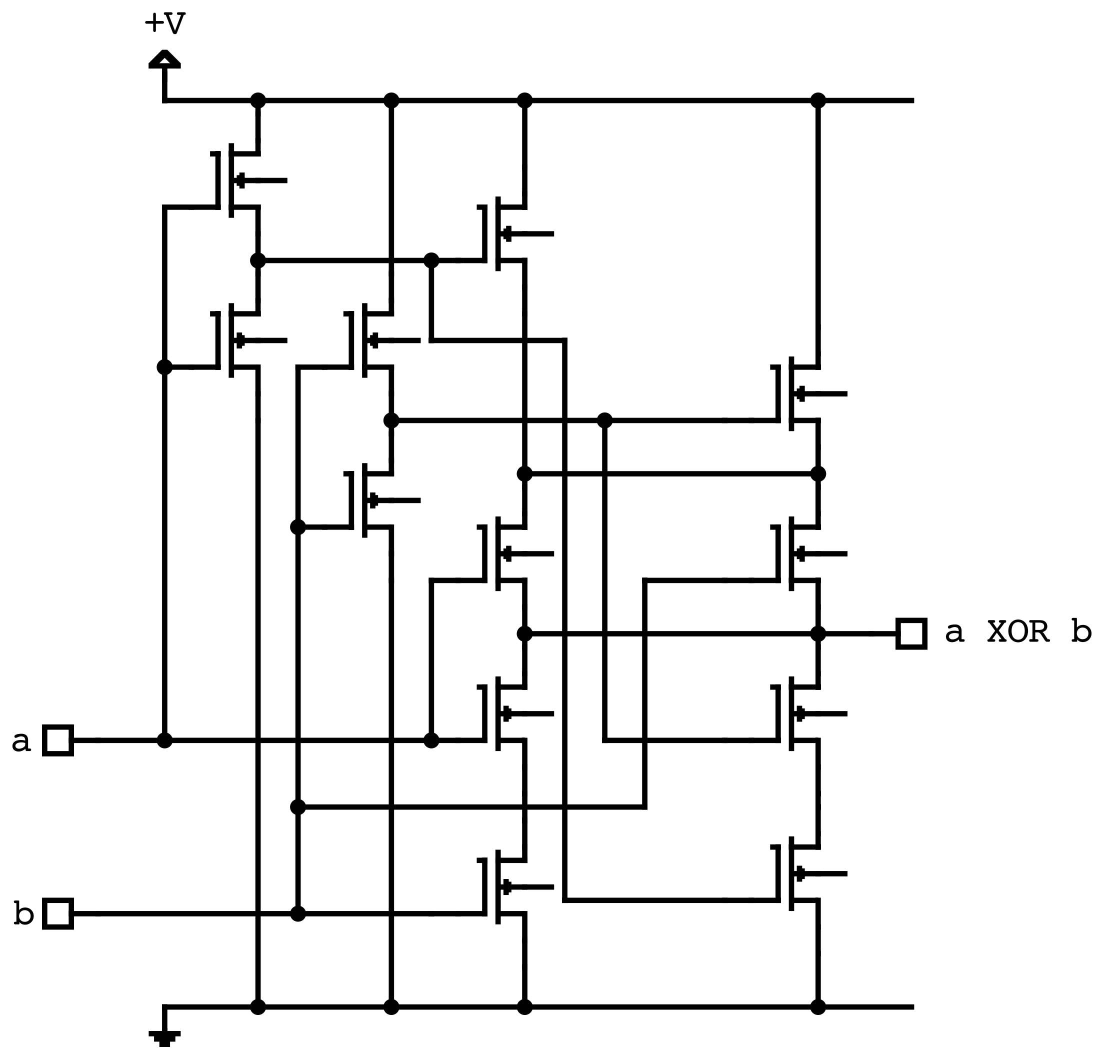
Circuit Schematics










+V
a
b
a XOR b

DIAGRAMMATIC PROOF FAMILIES

The following are proof families of representations

Boolean Cubes
Circuit Schematics

Parens
Enclosing Circles
Distinction Networks
Distinction Steps
Distinction Rooms
Distinction Blocks
Bar Graphs
Distinction Paths

AXIOMATIC EQUIVALENCE

The various sets of axioms are all equivalent in that they all express what we have been calling primary logic. We have selected the parens form using the computational axioms of boundary logic to demonstrate a common basis of them all. Below, the *modus ponens* theorem (axiom) of conventional logic is proved using each of the spatial representations. We assume that only their computational axioms will be used.

AXIOMS

```
OCCLUSION       (( ) A) = <void>
PERVASION       A (A B) = A (B)
INVOLUTION      ((A)) = A          (actually a theorem)
```

PROOF OF MODUS PONENS

```
( ((a) ((a) b)) ) b = ( )

(a*(a'+b))'+b

NOT (a AND (NOT a OR b)) OR b
```

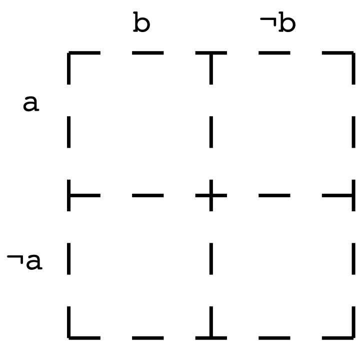
Parens

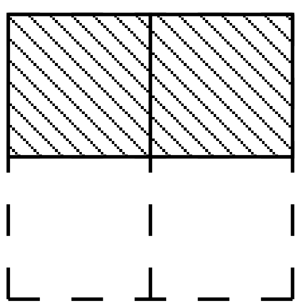
```
( ((a) ((a) b)) ) b           transcription
   (a) ((a) b)    b           inv
   (a) (      )   b           per (a) b
       (      )               dom
```

Boolean Cubes

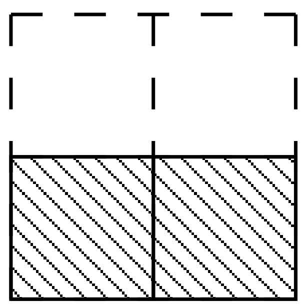
Constructive

a
NOT *a*
(NOT *a* OR b)
(*a* AND (NOT *a* OR b))
NOT (*a* AND (NOT *a* OR b))
NOT (*a* AND (NOT *a* OR b)) OR b

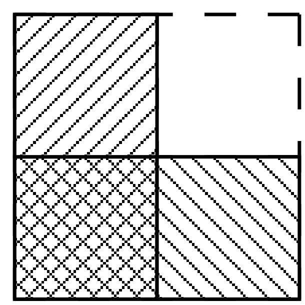




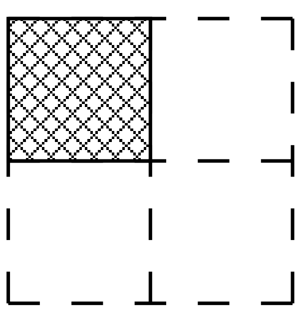
a



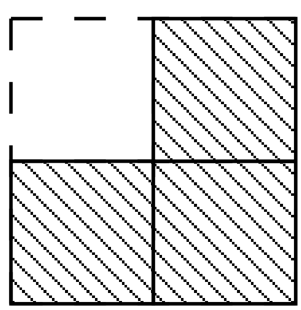
¬a



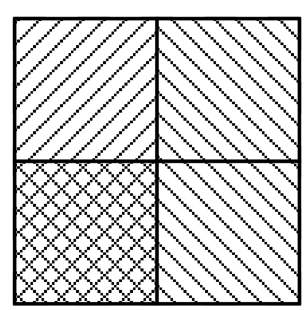
¬a **v b**



a & (¬a v b)

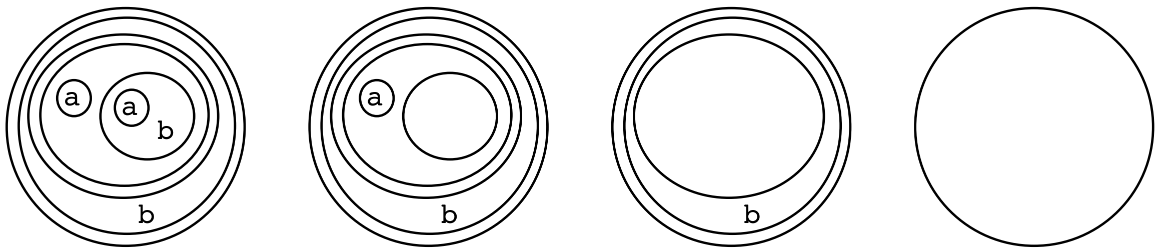


¬(a & (¬a v b))

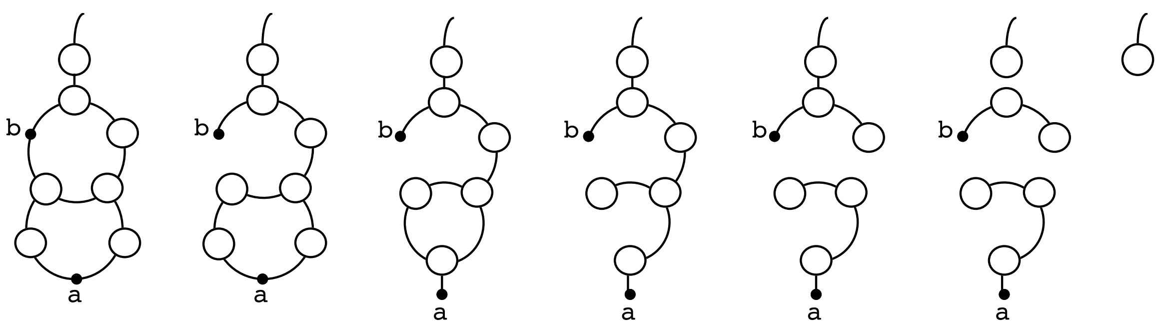


¬(a & (¬a v b)) **v b**

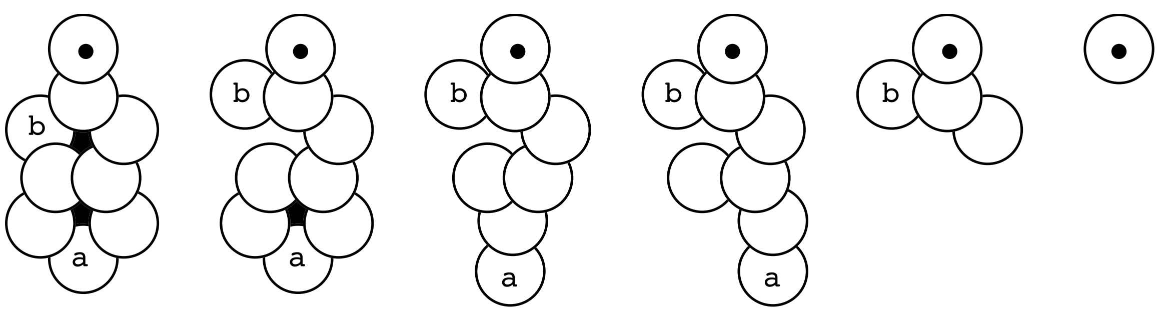
Enclosing Circles



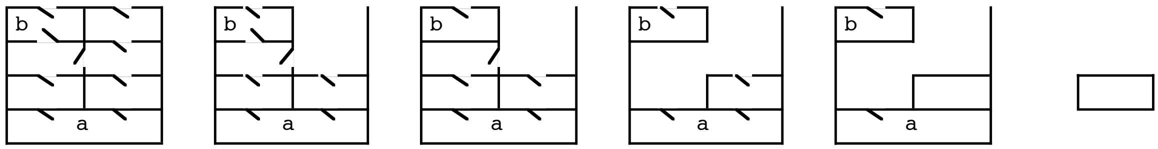
Distinction Networks



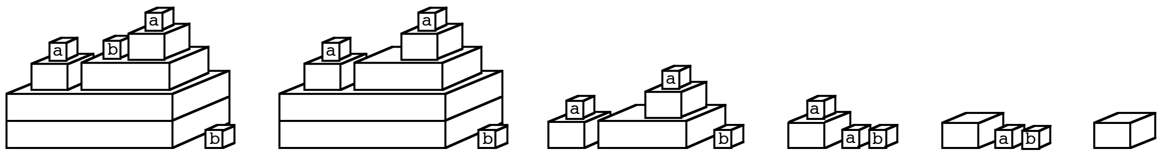
Distinction Steps



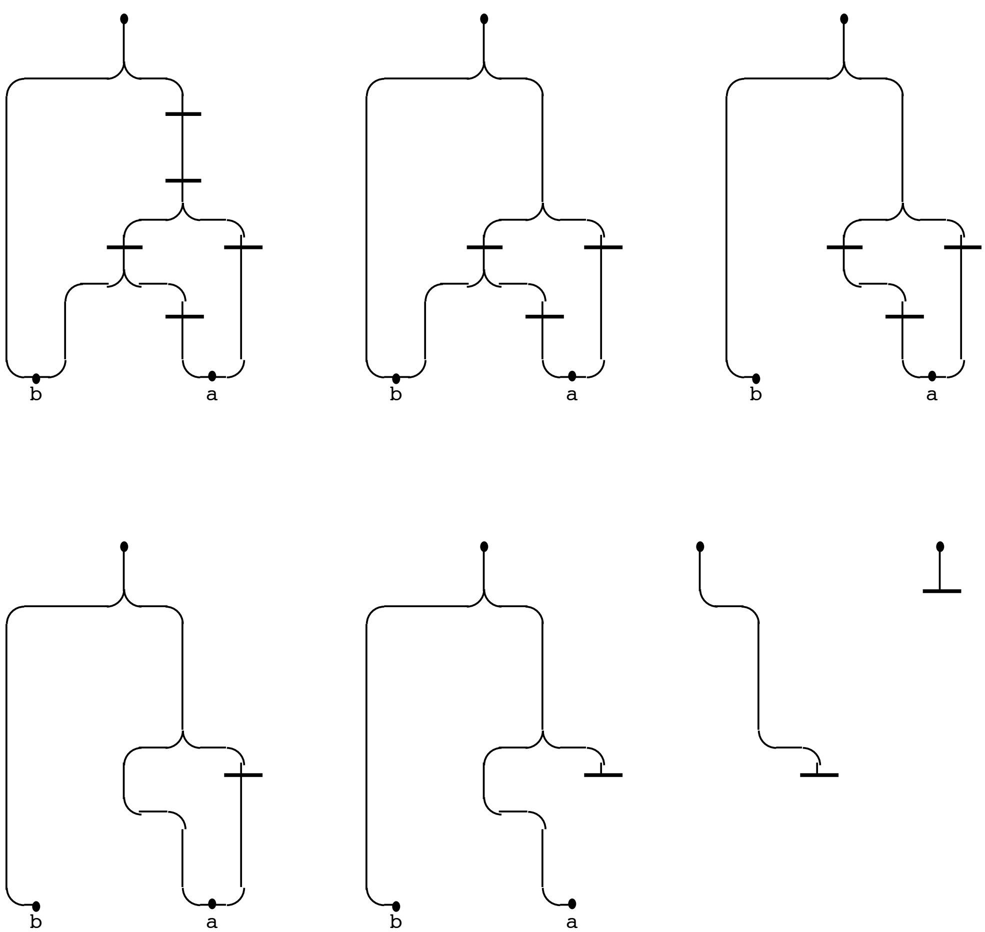
Distinction Rooms



Distinction Blocks



Bar Graphs



Distinction Paths