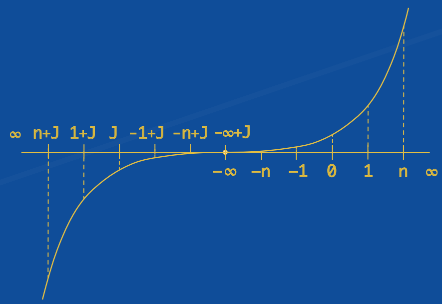
Iconic Arithmetic

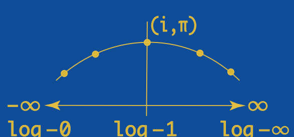


[ ][ ] ⇒ [ ]
( )[ ] = [ ]
( )( ) ≠ ( )
J J = *void*

simple

sensual

postsymbolic



Volume III

The STRUCTURE of IMAGINARY and INFINITE FORMS

William Bricken

Iconic Arithmetic

Volume III

Any comments, corrections, refinements or
suggestions you may have will be greatly appreciated.

You can reach me via email at
william@iconicmath.com

Thanks.

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Iconic Arithmetic

simple

sensual

postsymbolic

Volume III
The STRUCTURE of IMAGINARY
and INFINITE FORMS

William Bricken

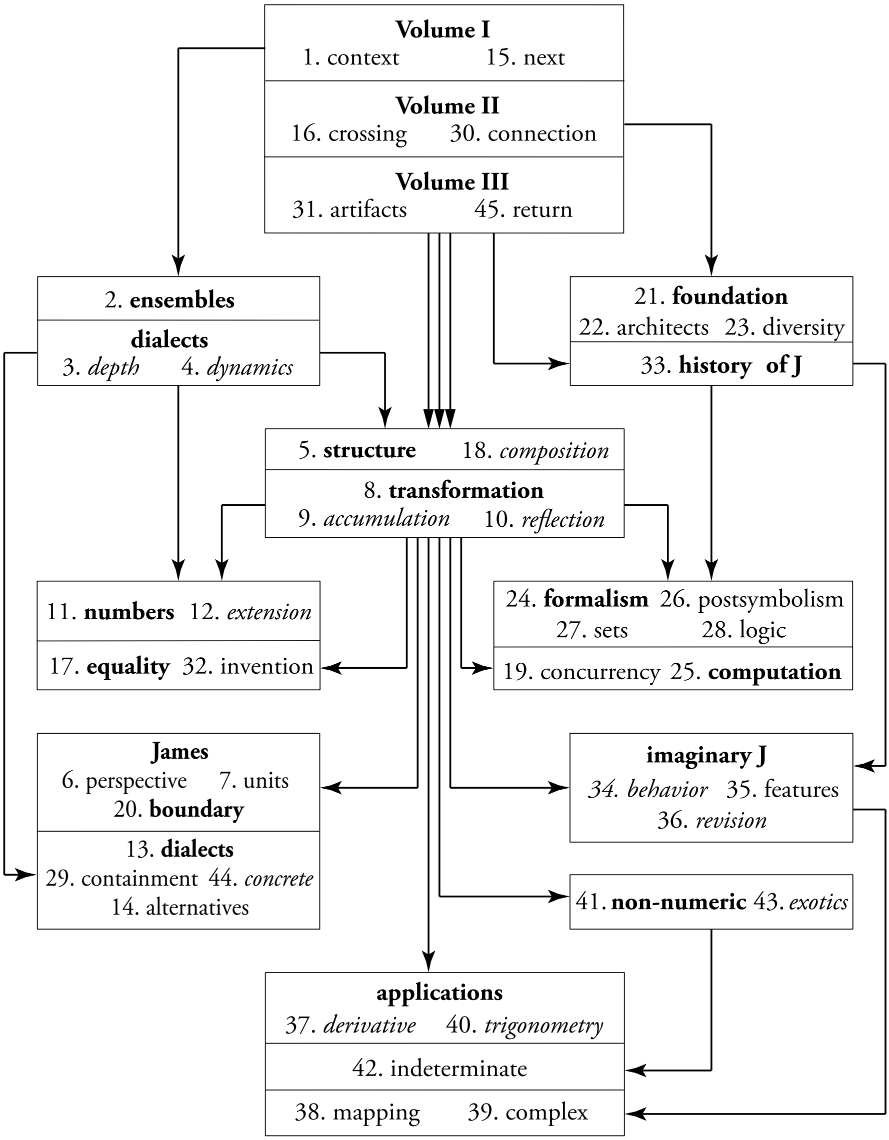
in memoriam
Richard G. Shoup
George Spencer Brown

Chapters Volume I

Chapters Volume II

Chapters Volume III

Chapter Map



Arrows indicate flow of content.
Bold font indicates the focus of the block of chapters.
Italic font indicates a focus on demonstration of transformations.

Contents Volume III

List of Figures

Cast of Characters

Visionaries

Aristotle
Gregory Bateson
John Horton Conway
Leonhard Euler
David Hilbert
Gottfried Leibniz
Benoit Mandelbrot
Charles Sanders Peirce
George Spencer Brown
Francisco Varela
Stephen Wolfram

Voices

Johann Bernoulli
Augustus DeMorgan
Leonhard Euler
Jeffrey James
Louis Kauffman
Gottfried Leibniz
Alberto Martínez
Charles Sanders Peirce
Richard Shoup
George Spencer Brown
John Stillwell

Research and Perspective

Gregory Chaitin
Louis Kauffman
Morris Kline
Barry Mazur
Joseph Mazur
Tristan Needham
John Stillwell
Gerald Weinberg

Colleagues

George Burnett-Stuart
Arthur Collings
Andrew Crompton
Graham Ellsbury
Jack Engstrom
Thomas Etter
Fred Furtek
Robert Horn
Jeffrey James
Louis Kauffman
David Keenan
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Jaron Lanier
Amy Morrison
Ted Nelson
Daniel Shapiro
Richard Shoup
John Walker
Stephen Wolfram

Historical Figures	***circa***	***chapter***
Anaximander	500 BCE	41
Pythagoras	500	36, 40, 41
Plato	400	40
Aristotle	350	38, *41*
Euclid	300	*43*
Bhaskar	1150 CE	41
Descartes	1640	33, 38, 40
Pascal	1680	*open-41*
Leibniz	1700	32, *33*, 42, 43
Johann Bernoulli	1720	*33*, 34
Newton	1720	33, 41-43
Euler	1760	*32*, *33*, 34, 35, *36*, 37, *38*, 39-42, 44
Gauss	1840	39, *41*
DeMorgan	1860	*33*, *39*
Riemann	1860	41
Cantor	1880	41
Hilbert	1900	31, *41*, 45
Peano	1900	32, 34, 41
Peirce	1900	39, 45
Poincaré	1900	*41*
Wittgenstein	1940	41, *42*

Quotes (in italics) **and Concepts**	***chapter***
Antonin Artaud	*open-37*
Alain Badiou	*45*
Deepak Bal	*33*
John Barrow	*open-32*
Claude Bernard	*open-42*
János Bolyai	*open-33*
Jorge Luis Borges	*41*
David Bressoud	*open-43*
Gerolamo Cardano	*33*, 39
Gregory Chaitin	41
Winston Churchill	*open-35*
Alain Connes	*41*
John Horton Conway	*34*, 41, 43

Mentioned *chapter*

Preface

For the purposes of communication, a choice has to be made,
but such a choice necessarily limits the presentation
and ignores the unlimited delights of the exploration itself.
— George Spencer Brown (1996), Introduction to Appendix 4: An algebra for the natural numbers, Laws of Form revised fifth English edition (2009) p.118.

Well, the end of a seven year project. I recently met a correspondence friend for the first time. He immediately asked: "Why can't you write a *small* book?" My honest reply was that I had tried to write for an audience, and found that it was a skill not within my reach. So I'm writing to meet my personal goals and standards, and for consistency with a couple of (um, perhaps risky) decisions made early. The goal is to show the simplicity of iconic arithmetic with an abundance of examples in order to convey a single message: the entire content of school mathematics can be described by a few definitions that anchor how iconic constants might behave, supported by three visually simple pattern-recognition axioms that animate the behavior and transformation of James forms. The half-dozen useful theorems are simple combinations of the axioms, usually standing in place of no more than four or five substitution steps. Volume I seeks to describe and to illustrate the visual dynamics of James algebra while emphasizing the conceptual beauty of void-based thinking and thus to motivate a completely different way of thinking about and understanding elementary arithmetic. The meta-goal is also simple: to demonstrate that *symbolic mathematics* is a choice rather than a necessity by providing a viable iconic alternative.

The first volume went smoothly and relatively quickly given the hundreds of accompanying illustrations. The goal of Volume II is to relate iconic thinking to the evolution of metamathematics and to the revered contributions of the founders of modern axiomatic arithmetic. That volume developed significantly slower since it involved substantial and substantive research. The seminal work from over a century ago had not yet clearly formulated fundamental ideas about meaning and structural transformation. The formal concepts being explored were also (obviously) pre-computational and pre-electronic. I struggled with antiquated motivations that were simultaneously prescient and anachronistic. Volume II is a journey into feeling inadequate yet opinionated. The primary challenge was learning how to see through the nearly universal acceptance of the axiomatic structure of sets and logic and whole numbers to find underneath an elegant, postsymbolic alternative.

This volume though has been the most challenging of the three, for mainly human reasons. Many technical details needed to be refined in order to expand James algebra into topics that were not originally intended but do serve as useful application examples. The goal is to present each application domain from an innovative, postsymbolic perspective. While trying to finish the first volume I elected to put aside nearly completed explorations until the "next volume", a decision directly caused by, yes, too many words. While updating the "nearly finished" content from 2015 and on, I was fully expecting an editing and compilation task. What I found was a whole lot of redundancy. Over the years the same ideas were repeated many times in many different contexts. My understanding had evolved, my errors had moved from one dimension to another, complicated dilemmas turned simple while new tangles were exposed, and worst of all, integrating a two-foot stack of new notes into already written words was a nightmare. I had thought that what *was* could be converted into what is, while holding a totally inappropriate belief that time is in short supply. Truth is, time doesn't care. And then 2020 pushed back and yet another year passed without completion.

Volume III is an exploration of the interplay between formal symbolic knowledge and void-based iconic innovation. The sections on history, the imaginary `J`, applications of James algebra and non-numeric forms are relatively independent. The heart of arithmetic consists of three concepts: zero, one and infinity. The heart of James algebra embraces three grounding concepts: the nonexistent *void*, the accumulator `( )` and the unifier `[ ]`. This volume begins with a resurrection of Euler's work on the logarithms of negative numbers in the form of the numeric constant `[<( )>]`, abbreviated as `J`. The first three chapters explore the history, behavior and issues associated with `log -1` and its apparent neglect in modern arithmetic. The next three chapters examine the intimate relationship between π, `i` and `J` established by Euler. Then follows a collection of applications of postsymbolic formal thinking to selected subfields of elementary mathematics. If iconic arithmetic is more than an isomorphism, then its application should suggest entirely different perspectives on established mathematical systems. These four chapters are unabashedly exploratory, recounting experiments, dead ends and forced structural conclusions. The next three chapters wrestle with the role of `[ ]`, a fundamental "unit" within the James arithmetic that does not accumulate and in that sense is non-numeric. Some call it infinity. The final content chapter in this volume revisits the spatial dialects of Volume I.

This volume opens the door to several partially explored radical re-visions of the concepts of numeric arithmetic, including

— arithmetic without addition and multiplication
— quasi-numeric illusions that create the impression of complexity
— the additive imaginary `J` as a numeric unit
— a non-distributive constant
— sign-blind and bipolar numbers and operations
— quantized fractions of polarity
— `e` as defined by the transparency of `( )`
— a generic form of the derivative
— `i` as a compound imaginary
— reflection as a foundation for rotation
— trigonometry without memory
— the non-numeric infidel `[ ]` and its reflection `<[ ]>`
— organized indeterminate expressions
— exotic powers and bases, and of course
— the ever absent *void*.

The postsymbolic dialects of Volume I are constrained in Volumes II and III to the convenience of the **parens** dialect, a typographic yet still iconic notation. At the cost of introducing some unintended symbolic artifacts (called *accidents* in the text) such as reading "`(`" and "`)`" as different tokens, bracket notations allow a concise visualization of the dynamics of James transformations by enlisting successive lines of the page to display temporal evolution.

The side columns on each page hold handy illustrations and reminders in support of the text. In the case of formal transformation sequences, the rules being applied are listed line-by-line in the sidebar.

All structural necessities for understanding James algebra are included in Figure 31-1 of Chapter 31.

In reading the text, backward reference to Chapters 1 through 15 refer to Volume I, while Chapters 16 through 30 refer to Volume II.

All references to online content have been verified as accessible during mid-2020.

The `iconicmath.com` website is the nexus for the content in these volumes and for potentially forthcoming volumes focused on computational logic.

The notation within the three volumes is consistent but does include a few symbolic characters that have unusual roles.

I've used typographic delimiters, `( )`, `[ ]`, < > and others rather than Spencer Brown's spatial mark, ⌉ , for easier typography and to make available several different representations for types of spatial containers.

A fixed width Monaco font identifies mathematical forms and functions, while the linguistic narration and discussion, the metalanguage so to speak, is printed in Cochin font.

The finger ☞ indicates a change in formal system, usually transcription between iconic James forms and conventional string expressions.

The numeric unit represented by a round-bracket has two forms, `( )` and `o`.

The arbitrary James base is represented by #.

The quasi-token *void* is meant not to exist.

A *frame* is the recurrent James structure `(A [B])` with `A` the *frame type* and `B` the *frame contents*.

Should you find typographic and/or conceptual errors in these pages, please freely contact me at `william@iconicmath.com` with corrections, suggestions and general discussion.

Take care.

william bricken
Snohomish Washington
December 1, 2020

Iconic Arithmetic

Volume III

Chapter 31

Artifacts

Who would have thought that almost everything, in the vast world of mathematics, follows from a few basic facts?[1]
— John Stillwell (2016)

We are exploring James algebra and in the process discovering the structure of an iconic formal system. The objective is to experience new methods of formal thinking rather than to discover new mathematical truths. Rearranging symbolic expressions is more efficient than redrawing boundary forms by virtue of hundreds of years of evolution. But do symbolic systems engender particular habits of thinking? Which cognitive processes does symbolic thinking discourage? Does memorizing rules for symbol manipulation constrain visualization of images or intuitive understanding or behavioral flexibility? Boundary numerics bridges two worlds, the ancient and the electronic. Ensemble arithmetic and James form are much closer to the physical interactive mathematics prevalent Before the Common Era. With fewer types of transformation, with atomic steps taken in parallel, and with the power of void-equivalence, boundary math also resembles optimized computer architectures.[2] Coupled with multisensory interaction, iconic methods encourage **embodied cognition** by removing the barrier between representation and meaning.

31.1 Pre-computational Thought

The apparent simplicity of the arithmetic and algebra of numbers is an artifact of a particular type of processing architecture that may be characterized as **pre-computational human symbolic abstraction** from about 1830 to 1980. Prior to that period numbers were more magic than skill-based. After that numbers adopted a binary format suitable for newer, more efficient silicon architectures. During the 150 intervening years culture and technology underwent profound changes, changes that have only presaged the greater transformation that humanity is now experiencing. The last half of last century endured an awkward transitional period during which our cultural expectation was that humans *need* to be able to process (as opposed to understand) symbolic numeric structure. In *this* century however it has become increasingly clear that humans need to understand how to use the exceptional silicon and algorithmic tools we have developed, regardless of whether or not we understand (or can understand) the 10^{18} transistor transitions per second from which a modern silicon computer generates multimedia experiences.

Educators have traversed the growth of technology with great trepidation. The debatable purpose of schooling is to acculturate as well as to educate. Educational bureaucracies are not designed for accelerated exponential change so much as for preservation of historical values. At the turn of the twentieth century, prior to both electric and silicon technologies and concurrent with the invention of universal schooling, the architects of curriculum for elementary education nominated memorization of arithmetic facts as a cultural and cognitive necessity. Group theoretic algebra was entombed as the *rules* of arithmetic and they too were deemed essential for an educated child to have memorized. As tools designed to replace memorization of facts and paper-and-pencil algorithms evolved, educators reluctantly gave up their belief that it is necessary for school children to be able to compute the square root of a number by hand.[3] Within decades after that, belief in the need to do long division by hand was abandoned. Multi-digit multiplication and even addition of large numbers are now hanging tenaciously by a thread.[4] However the dominant feature of classroom mathematics is that its functional content is *not* how modern relational computation is conducted, is not even close to the algorithms used in silicon computation and is not composed of skills needed for success in the 21st century.

Our extended exploration of iconic arithmetic and pattern-directed transformation has been motivated by fundamental cultural questions about mathematics education and about the tyranny of symbolic arithmetic. Is *symbolic numeric processing* a necessary skill for humans to have mastered by the time they reach their teens? Is *functional thinking* a necessary skill within a world that has moved decisively into relational and contextual values? Indeed is the universal embrace of symbolic algorithms appropriate for an age in which the majority of the Earth's population have in their hands or in their pockets a superb tool for parallel processing of patterns, whether they be numeric, pictorial, temporal or behavioral?

31.2 Adventure

Stepping aside from educational, technological and cultural considerations this journey into iconic arithmetic can be seen as an adventure, an *exploration of a foreign territory*. The role of an explorer is to visit new and exotic lands and to bring back stories and artifacts. An explorer of cognitive realms, whether they be philosophical or literate or mathematical or neurological or psychedelic, journeys inward, often in isolation, exploring non-physical territories and seeking to bring back conceptual artifacts that may be of interest to others. These volumes then are the diaries of a particular kind of adventure by a particular adventurer who is insufficiently acculturated to believe that formal knowledge is embedded exclusively within long strings of arbitrary tokens.

The territory is **formal cognitive distinction**. Formality assures stable and replicable artifacts. The formal structure of the explored territory is three algebraic pattern axioms, or beliefs, about how configurations of mutually nested containers can be transformed. The vehicle of exploration is pattern equivalence between forms of containment. The found artifacts brought back are patterns, or theorems, that depend upon both the structural constraints imposed by the axioms and their medium of expression. For a cognitive explorer to avoid traps and chasms within the imaginary realm a *map* is extremely valuable. For this exploration the map is from iconic distinction to the structure and behavior of symbolic arithmetic. Found artifacts are thus anchored by formal constraint, by conventional familiarity and by the shape of their forms.

Prior to the James axioms and in an attempt to characterize the natural numbers, Volume I postulates *unit ensembles* that support both containment and accumulation. The strategy was to begin with the mind of a preschool toddler to build patterns of discrete units. The spatial patterns themselves mimic the operations of arithmetic with forms that can be manipulated by both hand and eye. The postsymbolic containment patterns enabled by the James form provide a simple way to represent and to understand symbolic numeric expressions. But to see the simplicity naturally requires familiarity with iconic formal thinking.

What stands out within iconic form is the absence of many old and familiar numeric artifacts that have been assumed to be indispensable. There are, for example, no signs that identify the operations of arithmetic and no independent concepts of addition or multiplication. There is a general rejection of real numbers. There is the postsymbolic unification of syntax and semantics. There is in fact sufficient departure from modern symbolic mathematics to consider iconic form to be a different species of formal thought.

Treasures

Volume I focuses on visual and manipulable structures. Volume II takes an historical side trip to recount and compare the artifacts found by the first explorers of modern formal numerics about 140 years ago. Many of their discoveries have been canonized today as *what numbers are,* even though these artifacts were mined from cognitive environments that did not include vitally relevant experience with electrical machines, silicon computers, a global internet and a universe of software applications. Particularly absent from the cognitive artifacts of a century ago are modern knowledge engines such as *Mathematica* that can integrate sound, images, videos and databases within computational systems. Particularly absent from the mathematical artifacts of yesteryear is postsymbolic structure that reaches across the visceral territories of embodied thought.

Volume II explores iconic form with an emphasis on equivalence relations over patterns with a numeric interpretation. The conceptual artifacts brought back include void-equivalence, base-free exponents, dynamic objects and structural minimality. Figure 31-1 is repeated from Figure 16-1 of Volume II. It is the velvet cloth upon which are displayed both found treasures and cherished beliefs. The technical names of these structures are signposts installed along the way for others to follow. The new methods of iconic thinking are grounded by their *comparison* to well known methods of symbolic thinking. These new perspectives are the spoils of discovery. Their value is moot. Hopefully the iconic approach has sufficient value in its novelty, in its unexpected forms and in its anchors to symbolic concepts.

The Edges

Separated from the iconic forms of Volume I and from the contextualization of these forms in Volume II are the more exotic artifacts of this volume. Exotic conventional objects such as -1 and $\sqrt{-1}$ and ∞ have been studied for hundreds of years. Mathematicians now have excellent

Axioms and Theorems of James Algebra

Ground Interpretations

o = ( )	☞	1	(o)	*=def=*	#
< >	☞	0	<o>	☞	–1
[ ]	☞	–∞	<[ ]>	☞	∞ *(volume III)*

Unit Definitions

( ) ≠ *void*	**existence**	
( ) ( ) ≠ ( )	**unit accumulation**	
[ ] [ ] ⇒ [ ]	**unification**	*(volume III)*
[ ]<[ ]> ⇒ *indeterminate*	**indeterminacy**	*(volume III)*

Pattern Axioms

([A]) = [(A)] = A	**inversion**	*enfold/clarify*
(A [B C]) = (A [B]) (A [C])	**arrangement**	*collect/disperse*
A <A> = *void*	**reflection**	*create/cancel*

Interpretative Axiom *(volume III)*

(<[ ]>) = <[ ]> = [<[ ]>]	**infinite interpretation**

Theorems

( ) <( )> = *void*	**unit reflection**	*create/cancel*
([ ]) = [( )] = *void*	**void inversion**	*enfold/clarify*
(A [ ]) = *void*	**dominion**	*emit/absorb*
A = ([A][o])	**indication**	*unmark/mark*
A. .$_{N}$. .A = ([A][o. .$_{N}$. .o])	**replication**	*replicate/tally*
<<A>> = A	**involution**	*wrap/unwrap*
<A><B> = <A B>	**separation**	*split/join*
<A <B>> = <A> B	**reaction**	*react/react*
(A [<B>]) = <(A [B])> (A <[<B>]>) = <(A <[B]>)>	**promotion**	*demote/promote*

Figure 31-1: *Summary of definitions, axioms and theorems (Figure 16-1)*

strategies for dealing with, for example, complex numbers and infinite sequences. The three James pattern axioms were initially chosen to shed light upon natural numbers. From them arose two creatures that are so fundamental yet so unexpected that they deserved a volume of their own. The first, [<o>], labeled J, was discovered as a symbolic expression three hundred years ago and then lost in obscurity until now. This volume tells its story. The second, [ ], what we are calling the *square unit,* is non-numeric yet embedded at the very foundations of the iconic territory defined by the axiomatic belief system.

This volume also includes brief explorations of the derivatives of functions, the structure of trigonometric forms, the construction of imaginary numbers and the varieties of non-numeric form that we call infinity. These inquiries are experiments in the application of the James algebra to some well-known topics in elementary mathematics. The volume ends by returning to the visceral iconic dialects in Volume I.

The Structure of Numbers

Numbers are widely described as belonging to nested subsets. It is tempting to believe that natural numbers ($\mathbb{N}$) give rise to integers ($\mathbb{Z}$) which generate the rationals ($\mathbb{Q}$) which evolve into the reals ($\mathbb{R}$) which are somehow directly connected to the complex numbers ($\mathbb{C}$). Expressed symbolically as *containment relations,*

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Figure 31-2 provides the definitions of these double-struck symbols for number types.[5] There are certainly elements of group theoretic structure in these categories. Whole numbers are natural numbers with the additive identity 0 appended. Integers append the additive inverse $-n$. The multiplicative identity 1 is embedded as the foundation of natural numbers. Rational numbers append the multiplicative inverse $1/n$. Irrational numbers have been

number type	***includes***	☞	***James algebra***
ℕ *natural*	n		o o ≠ o
whole	0		*same as natural*
ℤ *integer*	−n		<ℕ> = (J [ℕ])
ℚ *rational*	1/n		(<[ℤ]>) = ((J [[ℤ]]))
ℝ *real*	*order only*		[ℚ]
imaginary	bi		(J/2 [ℝ])
ℂ *complex*	a + bi		ℝ (J/2 [ℝ])

Figure 31-2: *The structure of conventional and James numbers*

Richard Dedekind
1831-1916

known since the ancient Greeks however until Dedekind defined the real numbers in 1888. Irrationals such as √2, π and e, were generally limited to numbers that had a distinct geometric meaning. **Algebraic numbers** are roots of polynomial equations, their category is uniquely linked to the historical evolution of algebra via a fascination with solving polynomial equations during the European Renaissance. Algebraic numbers are chimera, composed of incommensurable number types such as a natural number and an irrational, 1 + √2 for example. As mentioned in Chapter 27.2, the real numbers have the dubious distinction of being overwhelmingly composed of *lawless numbers* that are both indescribable and unknowable. The complex numbers, such as 3 + 4i, are a special type of algebraic number composed of two real numbers, one multiplied by the imaginary i.

Conventional numbers are an historical and evolutionary aggregation of concepts and design decisions. Each type is a *closure* that assures that operations on the type stay within the bounds of that type, with various exceptions of course like divide-by-zero. Figure 31-2 compares these traditional categories to the structures elicited by the James axioms. The James analogs are only examples and are not comprehensive. In particular James algebra does not include all real numbers. Since logarithms are transcendental irrationals, [Q] does include *describable* real numbers.

The James design decisions result in a completely different answer to the question: *What is a number?* James algebra trades our conventional diversity of notation for a minimal notation and much simpler operations, leading to a significantly more coherent design approach with

— *no zero*
— *one constant* J that is a focus of this volume
— *one operation*: containment
— *two types of container*: round and square[6]
— *three pattern axioms*[7]
— *one proof method*: pattern substitution
— *three exceptional rules* to handle non-numbers.

The James iconic pattern transformations cover all of elementary mathematics yet do not include the operations of addition, subtraction, multiplication or division.

addition
A + B ☞ A B

subtraction
A – B ☞ A (J [B])

multiplication
A x B ☞ ([A][B])

division
A/B ☞
([A](J [[B]]))

power
B^A ☞
(([[B]][A]))

root
$B^{1/A}$ ☞
(([[B]](J [[A]])))

logarithm
$\log_B A$ ☞
([[A]](J [[[B]]]))

31.3 Remarks

The many annotated demonstrations in this volume cover imaginary logarithms, differentiation, trigonometry and infinite forms. Collectively they support a primary observation that James forms are sufficient to construct almost all of elementary arithmetic. Harvey Friedman and many others who have recently developed **reverse mathematics** established that *arithmetic is sufficient* for constructive mathematics, a goal set by Hilbert as the **arithmetization of mathematics**.[8] The highly technical work on *computable foundations* suggests that the iconic exploration of diverse mathematical domains is of interest both theoretically and pragmatically. This volume is too primitive to make substantive contributions to either the theory or the pragmatics of computation. *Iconic Arithmetic* instead attempts to shine *first light* into iconic territories that are generally unexplored. We'll now step back to find the origin of J.

Endnotes

1. **opening quote:** J. Stillwell (2016) *Elements of Mathematics* p.8. Beginning with *almost everything*, the original is also in italics.

2. **boundary math resembles optimized computer architectures:** The RISC computational architecture is a Reduced Instruction Set Computer. The processing unit uses very few instructions but does them a lot of times.

3. **compute the square root of a number by hand:** As I did as an Australian 7th grader in the 1950s.

4. **large numbers are now hanging tenaciously by a thread:** Addition of columns of four digit numbers is still thriving, as is chanting the numeric matrices that structure digital addition and multiplication. Analogous content in English class would return us to the eighteenth century for memorization and recital of epic poems written in Middle English.

5. **definitions of these double-struck symbols for number types:** The origin of the double-struck typography appears to be very recent (within the last sixty years) as is the entire idea of agglomerating numbers into subsets. Retrieved online 7/20 `https://mathworld.wolfram.com/Doublestruck.html`

6. **types of container: round and square:** In this volume, the *angle-bracket* is taken to be an abbreviation for the constant `J` expressed within a J-frame, `(J [A])`. The two ensemble grouping axioms from Chapter 3.2 provide an additional type of bracket that pragmatically integrates into the James form a grouping notation for large accumulations of units that is the equivalent of adding columns to our conventional notation.

7. ***three pattern axioms***: The pattern shared across these axioms is the James frame `(A [B])`, described in Chapter 35.1. Three additional axioms that characterize the behavior of the non-numeric square unit `[ ]` extend the James algebra into forms associated with the concept of infinity.

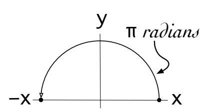
8. **by Hilbert as the arithmetization of mathematics:** Reverse mathematics asks which axioms are best suited for supporting the diversity of mathematical theorems. An excellent introduction to the field is J. Stillwell (2018) *Reverse Mathematics: Proofs from the inside out*.

Chapter 38

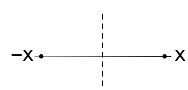
Mapping

Words and pictures belong together.[1]
— Edward Tufte (1983)

We have explored J as an imaginary numeric concept. Euler defined J to be a conventional imaginary number with an infinity of values $\pm(2k + 1)i\pi$ for the exponent of the function e^z. The appearance of $i\pi$ indicates a *rotation* in the complex plane and designates the exponent z to be a complex number. We will explore complex rotation and Euler's formula in the next chapter. Here we'll consider J to be a logarithmic *reflection* rather than an imaginary rotation and attempt to avoid both complex numbers and infinities of values. The form of J rather than $i\pi$ provides the power.



rotate



reflect

$$\#^J = -1 \qquad\qquad J = \log_\# -1$$

Euler considered $\log -1$ to be rotation through 180° thus necessitating the complex plane as an alternative dimension through which to rotate. When we consider J to be a reflection the "imaginary" J lands on the real number line. -1 is the reflection of $+1$, but where does J itself fall? J appears when we apply the logarithmic transformation to negative real numbers, very similar to the appearance

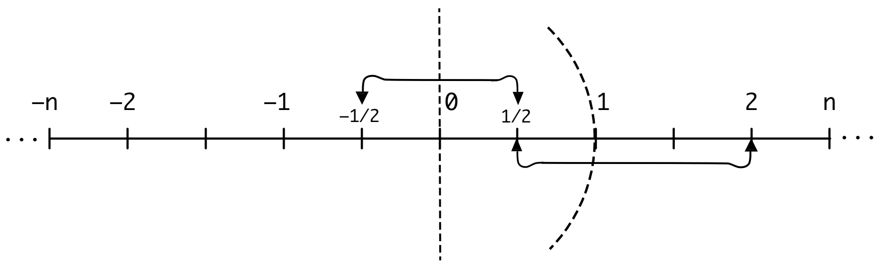
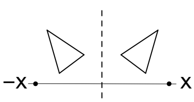


Figure 38-1: *Reflection through line and circle*

of `i` when we apply the square root transformation to negative numbers. Thus the form of `J`, `[<o>]`, can be read within our interpretation literally as the logarithm (square-bracket) of the reflection (angle-bracket) of a unit (empty round-bracket).

38.1 Inversion

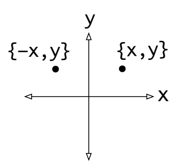


additive

In two dimensions reflection through a *straight line* is **additive inversion**. Negative expressions reflect positive expressions. An expression and its reflection mutually `cancel` by addition, taking both to the origin `0`.

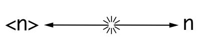
A ⟼ −A ☞ A ⟼ <A>

A + −A = 0 ☞ A <A> = *void*



infinite plane

The geometric concept of reflection through a line is usually presented within the context of a **plane** that consists of an infinite number of points, each indexed by a *pair* of real numbers. This permits any plane geometric figure to be reflected in its entirety by attaching a negative sign to each `x` value. Here we will explore the simpler case of individual real numbers reflected through a *point* on the number line. For familiarity we'll call that point `0`, although the James reflection point zero does not exist.

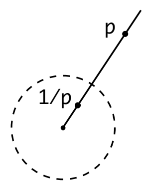


James number line

Reflection through a *unit circle* results in **multiplicative inversion**. The reciprocal of an expression reflects that expression. The two multiplied together mutually `cancel` to yield the radius of the circle, the unit 1. The James form below reduces via `unit cancel`, the same transformation mechanism that reduces an additive reflection.

```
A ⟼ 1/A        ☞    A ⟼ (<[A]>)
A x 1/A = 1    ☞    ([A][(<[A]>)])
                    ([A]  <[A]>  ) = ( )
```

The less familiar reflection through a circle provides a visualization of the relation between numbers and their reciprocals. Figure 38-1 provides an illustration. The study of the structural relations between points inside a circle and those outside a circle is called **inversive geometry**. Additive reflections are linear and support Euclidean geometry. Multiplicative reflections are non-linear and lead to non-Euclidean geometry (hyperbolic geometry).

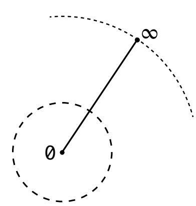


multiplicative

Inversive geometry defines the length of a unit to be the radius of the unit circle. Let P stand for the distance from the origin of a circle of radius r to an arbitrary point outside the circle. The associated point 1/P inside the circle is defined by multiplication of distances such that

$$P \times 1/P = r^2$$

Any point at infinity ($P = \infty$) reflects to the origin of the circle ($1/P = 0$). This suggests that there is a circle of points at infinity together forming the circumference of a circle with infinite radius.

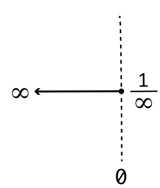


point(s) at infinity

On the circumference of the reflective circle, $P = 1/P$, which is to say

$$r \times 1/r = r^2 \qquad \textit{only when} \qquad r = 1$$

Reflection through a straight line is a special case of reflection through a circle for which the radius of the reflective circle is infinite and every point on the circumference is zero.[2] Thus additive reflection places the reflective line at zero while multiplicative reflection places the reflective circumference at one. The multiplicative inverse is **anticonformal**, it maintains the angles of any figure reflected though a circle but reverses the orientation.



infinite radius circle

Within our interpretation we also have two inverse types of **exponential mapping**,

exponential

$$A \longmapsto \#^{A} \quad ☞ \quad A \longmapsto (A)$$
$$\#^{\log_{\#} A} = A \quad ☞ \quad ([A]) = A$$

logarithmic

$$A \longmapsto \log_{\#} A \quad ☞ \quad A \longmapsto [A]$$
$$\log_{\#} \#^{A} = A \quad ☞ \quad [(A)] = A$$

inversion
([A]) = A

where an exponential expression and its logarithm combine to have no effect on their argument. We might call this **bracket inversion**, reflection through a pair of inversion brackets.

The three types of reflection, or inversion, are fundamentally different.[3]

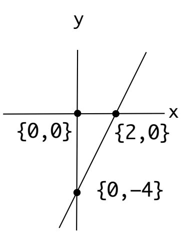
— **additive/subtractive**: linear reflection, flat geometry, *objects cancel* each other

— **multiplicative/reciprocal**: circular reflection, hyperbolic geometry, *square-bracketed objects cancel*

— **exponential/logarithmic:** functional reflection, iconic geometry, inverse *brackets cancel*

Here we'll take a closer look at how these *bracket reflections* interact with the structure of the conventional number line.

38.2 Plane Thinking

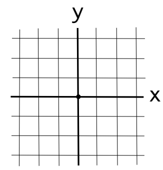


y = 2x – 4

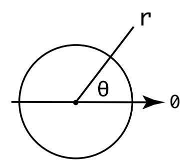
Descartes (along with many others of his time, as is always the case) saw how to convert algebraic equations, such as y = 2x – 4, into pictures drawn out on a flat plane with orthogonal x- and y-axes. The picture of an equation is a line drawn through all the coordinate points that are valid substitutions into the equation. This section introduces the Cartesian and polar perspectives for locating a point on a plane rather than on a line. These concepts will carry into the next chapter as we explore rotation and trigonometry, which will in turn allow us to explore the complex numbers that Euler found to be essential to understanding log –1.

Extrinsic and Intrinsic Perspective

There are many ways to identify the location of a point on a plane. Two of the most common are polar and Cartesian. The polar world-view is one of *orientation and distance* whereas the Cartesian world-view is of *absolute origin and orthogonal decomposition*. Both polar and Cartesian systems standardize distance to an arbitrary unit, defined as the distance between Cartesian unit grid marks and by the radius of a polar unit circle.

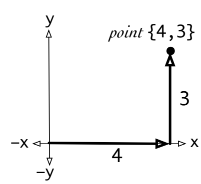


orthogonal



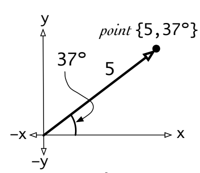
polar

We can identify any point on the Cartesian plane by naming its location with a pair of numbers, `{x,y}`, called *coordinates*. James notation uses round- and square-brackets, so to avoid confusion I'll use **curly braces** to represent a pair of coordinates.[4] The **origin** at the intersection of the two orthogonal axes has the coordinates `{0,0}`. The rectilinear, externalized infinite set of Cartesian coordinates make more sense as a way of thinking when we know where the origin is. To find the origin we must stand away from the coordinate plane by placing ourselves in a higher dimension. The **extrinsic perspective** pretends that we have god-like abilities — which we do have since we are viewing the coordinate plane from a third dimension — abilities that allow us to interact with the plane and its coordinate system externally and "objectively".[5]



orthogonal

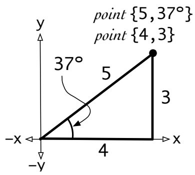
An alternative is the **intrinsic perspective**, the one that is natural for humans within a natural world. The intrinsic perspective makes us the origin. When we move, the origin moves with us, because the origin is anchored to the location of our direct perception. The use of direct distance and angle for coordinates is called the **polar coordinate system**. In place of the orthogonal `{x,y}` coordinates, we have the `{r, θ}` coordinates in which **theta**, θ, is the angle we turn through from our initial orientation and `r` is the direct distance from our location to the point in question. When we move toward the selected point our distance from it changes. In contrast



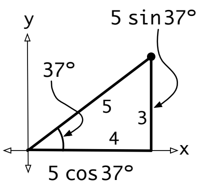
polar

the location of a Cartesian point does not change as we move, rather it is anchored by the origin. Here is the algebraic relationship between orthogonal {x,y} and polar coordinates {r, θ}.

orthogonal to polar	***polar to orthogonal***
$\sqrt{(x^2 + y^2)} \Rightarrow r$	$r \cos\theta \Rightarrow x$
$\arctan(y/x) \Rightarrow \theta$	$r \sin\theta \Rightarrow y$



right triangle



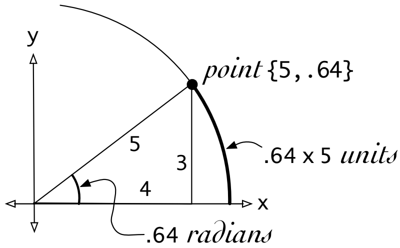
trigonometry

The first orthogonal-to-polar transformation is the Pythagorean theorem, $x^2 + y^2 = r^2$. It is common practice to discard the negative result of the square root function, under an assertion that distance is positive. This bias is a direct result of the extrinsic perspective, for which our location in the picture is taken external to the origin. From the intrinsic perspective our viewing location is mutable. The point {5,37°} is 5 units away from where we putatively stand. But it is −5 units away if we switch perspectives and consider distance *from* the point to our location, and 0 units away should we locate ourselves at that point. In general *semantic interpretation* is not objective. Whatever makes meaning to us is subjective. Objectivity presumes an externalized origin that in turn creates an illusion of detachment from meaning.

The second orthogonal-to-polar transformation is where trigonometry comes in. The trigonometric right triangle combines both Cartesian and polar perspectives. This is most apparent for the polar-to-orthogonal transformations. The sides of a unit right triangle are labeled sin and cos. Both depend upon the same angle θ. The trigonometric *functions* sine and cosine relate the (intrinsic) angle of rotation to the (extrinsic) x and y distances.[6]

Radians

We need one other piece of mechanism. There are two common but different ways to measure angles. Most familiar is the **degree system**, which defines 360 degrees as one complete revolution. This system is ancient, at

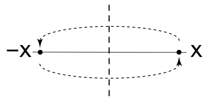


least 4000 years old. The alternative is to measure not the angle, but the distance traveled *along the circumference* of a unit circle. The **unit circle** is just a circle that has been scaled to have a radius the same size as the units 1 and i and J. The total distance of one circumnavigation is 2π, the circumference of the unit circle. Radian measurement does not depend upon the radius of the circle, it measures only a proportion of the 2π circumference of the unit circle.

The idea of traveling along the circumference of a circle is due to the ancient Greeks. Greek mathematics did not include our modern concept of measurement, it focused instead on relationships between geometric components. The degree system is *extrinsic*, we impose it upon a circle for our convenience. The radian system is *intrinsic*, it is inherently part of the nature of a circle. Our 90° rotation is to the ancient Greeks a journey 1/4 of the way around a circle, an *angular distance* of π/2.

38.3 Rotation as Reflection

Rotation itself has an inherent ambiguity. We can count the *number* of complete revolutions around a circle using memory, or we can begin again after each full revolution, never counting "how many". Rotation is taken to be strongly temporal, we usually do not imagine making many full rotations all at the same time. Rotation also encourages the idea that turning can be disassembled into angles with relatively infinite precision. This perspective comes from the association of angular measurement with a fraction of the circumference subtended by the **radian** measure of angle. Linear distance has been presumed to be continuously subdividable since Aristotle, as has the 360 angular **degrees** that we have inherited from the Babylonians. In contrast, *a reflection is discrete*.



If we limit "rotation" to only π radians, to only turning around or reflecting, then the idea of an angle vanishes entirely as would the process of rotating. The 180° reflection from +1 to −1 is characterized by a different type of transformation, binary rather than continuous.

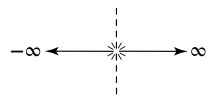
Geometrically we think of reflection as *across* a line, as if the line were a mirror. This line is sometimes even called the *mirror line*. To reflect values on one side of the x-axis to the other side, the mirror line is the vertical y-axis that defines the location $x = 0$. The x-axis extends from the origin in opposite directions to include both positive and negative values. This in turn confounds reflection with the concept of negation. Reflection is a change in *polarity*.

A reflection of an object from one side to the other followed by a reflection of the image from the other side back returns to the original starting point of the object. To maintain this symmetry, we establish that negative times negative is positive. This decision confounds addition of radians (rotation) with multiplication of signs (state change). Rotation in a plane also supports heading in one of two directions, clockwise and counterclockwise. The two are analogous to the negative/positive directions of the number line. Thus the concept of **orientation**, how we traverse a circle's circumference, is inextricably linked to the meaning of negation. It's necessary to make the relationship between rotation and negation explicit. Although consistent the relationship is not necessary. In the next sections we will indeed dissolve the association between rotation, reflection and negation.

J Reflection

In the James form numeric reflection across the x-axis (what is called negation) is replaced by the constant J. Algebraic negation and geometric reflection are formally both **involutions**, each is its own inverse. J however is an object as well as an operator. *Presence of* J *is not the inverse of its absence.* We are in conceptually different territory. As we shall see J identifies an imaginary extension of the logarithmic x-axis, "on the other side" of negative infinity.

James calculus has no zero, so there is a hole at conventional 0 on the James number line. Due to J-void, any even number of replicas of J can be found in that hole, as well as all other void-equivalent forms.[7]



Geometrically any rotation can be constructed from two reflections. Translation too is composed of two reflections. Reflection is the single primitive underlying geometric transformation in space. As a reflection without memory, J embodies no concept of orientation. J-transparency allows the real number line to be independent of the act of reflection.

J-transparency
[<(A)>] = A [<( )>]

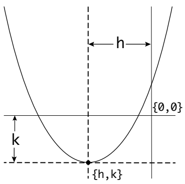
Algebraically we can reflect a form by placing it in angle-brackets. Angle-brackets themselves are shorthand for a **J-frame**.

A <A> = A (J [A]) = *void* **J-conversion**

We now need to abandon the idea of rotation until the next chapter and look instead at a one-dimensional line with a hole of nonexistence at the origin.

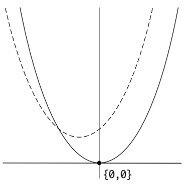
38.4 Drawing the Line

Geometry teachers make a point. Arithmetic teachers draw the line. They usually draw it on the walls, around the edge of the classroom above the windows where it can dominate both vision and thought. It is called the **number line**. Kids learn to chant the sequence: *"1, 2, 3, ..."* to infinity.



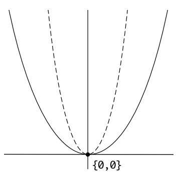
$y = a(x - h)^2 + k$

generic



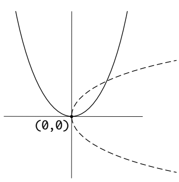
$\{h,k\} = \{0,0\} \Longrightarrow y = a\,x^2$

translate



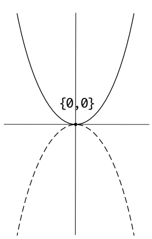
$y = ax^2 \Longrightarrow y = x^2$

scale



$x = y^2 \Longrightarrow y = x^2$

rotate



$y = -x^2 \Longrightarrow y = +x^2$

reflect

Students are then introduced to negative numbers as a reflection of the positives and to fractions stuffed inconveniently between the integers. Later we might learn that there are an infinity of unit fractions between `0` and `1`, reflecting the natural numbers as unit reciprocals.[8] In the process a learner might rediscover the ancient Greek idea that there is an outward infinity and an inward infinity, both on the same line and both without limit.[9]

Perspective

We can non-destructively translate, rotate, reflect and scale functions by *changing viewing perspective*. From the Cartesian perspective the origin and the coordinate axes are frozen as part of the symbolic representation of a function. The function itself then appears as a static line drawn within the Cartesian plane.

> *Symbolic expressions represent a rigid coordinate framework. Iconic forms represent generic intention.*

The common transformations of a two-dimensional figure are illustrated in the sidebar for an arbitrary parabola. These transformations can convert a generic quadratic function `F` into the simpler squaring function `f`.[10]

$$F(x) = (\pm a)(x - h)^2 + k \quad \longmapsto \quad f(x) = x^2$$

Self-multiplication and its inverse, the **square root**, are particularly important since they generate both sign-blind and imaginary numbers.

Quadratic Space

A single variable function `f` maps a relationship between values `f(x)` expressed along the `y-axis` to a line of values `x` along the `x-axis`. The function `f` usually curves through the two dimensions of an `x/y` coordinate plane, however the function itself is just another line in the plane, a one dimensional structure within a two dimensional space.

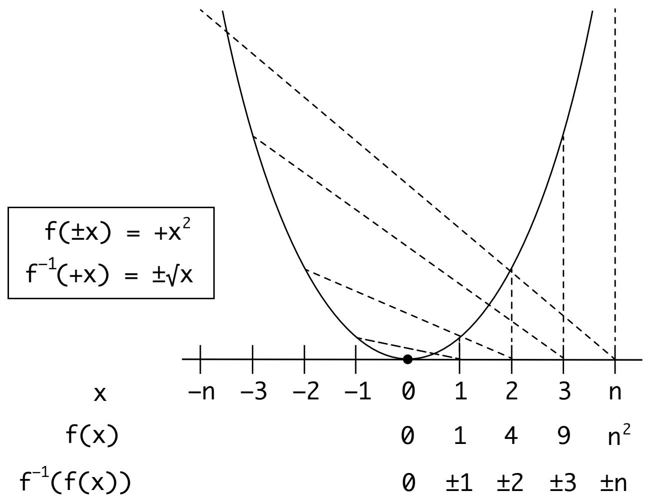
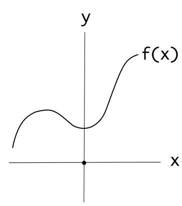
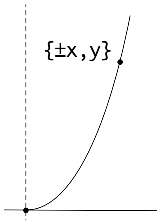


Figure 38-2: *Self-multiplication projected onto the* x-axis

We can map f(x) onto the linear scale of the horizontal number line by pairing the values on the vertical y-axis with the associated values on the x-axis. Figure 38-2 shows a graphic representation of self-multiplication, abbreviated as $f(x) = x^2$, over the entire range of positive and negative values of x. The spatial display of the y-axis in Figure 38-2 is *projected* downward onto the linear scale of the x-axis, in effect overlaying two sets of values on one line, one set structured as x and the other set structured as f(x). The f(x) scale is then a *recalibration* of the x-axis. In the figure of self-multiplication the size of the steps between unit x-axis values (i.e. the unit distance between the whole numbers) is now variable, each step is two units more than the prior step. For the projection of f(x) onto the horizontal axis, the distance from 0 to 1 is 1 unit; the distance between 1 and 2 is 3 units; 2 and 3 are 5 units apart, etc.



But what is particularly interesting about this function is that the original *negative* x values are all converted into positive values. The mapping $\pm x \longmapsto x^2$ also assigns all −x to their corresponding squared positive values on the



$f(x) = x^2$

number line. As we have previously observed, self-multiplication is sign-blind. This mapping is **two-to-one**, two x values are converted into one f(x) value. When the inverse square root function is applied to f(x), the mapping becomes one-to-two, both the positive and the negative values of x are resurrected, but now they are bound together as indistinguishably equivalent (*identical*) values of the square root function. One way to visualize this is to imagine that the left-side of the parabola does not exist, it is overlaid on top of the right-side while the x-axis is composed of bipolar numbers, ±x.

Birth of i

What then has happened on the left-side of 0, where the negative x values were? The well known answer is that they have retreated into our imagination. Figure 38-3 shows this finesse.

We create left-of-zero as the *imaginary numbers* by allowing self-multiplication to return in a negative numeric result.

$$x^2 = -n \qquad \textit{therefore} \qquad x = \sqrt{-n}$$

The imaginary domain is very much like Alice's *Through the Looking Glass* wonderland. If we consider the y-axis to be a mirror, we could say that the right side of the parabola is "real" while the left side is an image, an *imaginary reflection* of the real side. Although f(x) does not generate negative y values, we imagine them into existence by creating a *different kind* of "negative" x named xi. We give up numeric homogeneity along the x-axis by associating each point with two types of unit. The right side contains the real numbers and the left side contains the imaginary numbers. The left side inhabitants are designated by ni; they live in the imaginary ***i-land***.

What we have done (indeed what Euler did nearly three centuries ago) is not that strange. After all, the left side is already a different type of unit, "–" x 1. Only the label has been changed, to "i" x 1. Self-multiplication collapses +n

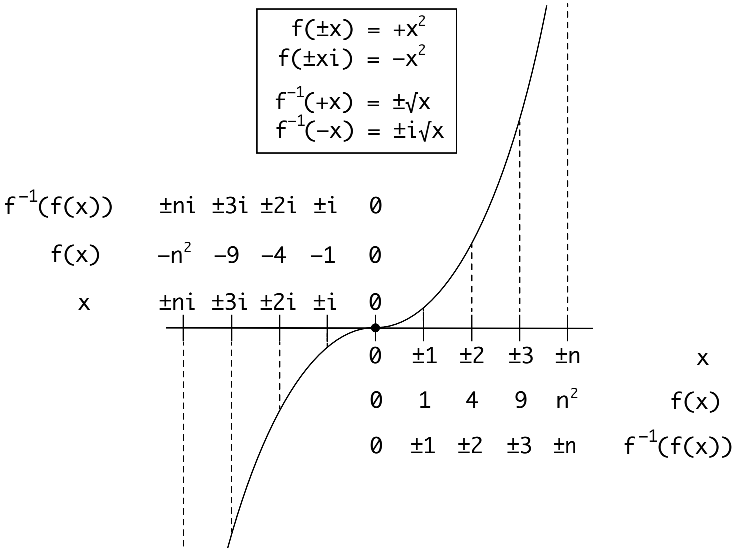


Figure 38-3: *Imaginary self-multiplication projected onto the* x-axis

and −n into ±n. The inverse square root function retrieves exactly that, ±n, from the values of the x^2. These bundled bipolar numbers describe how x^2 and $\sqrt{x}$ behave.

+n ⟼ ±n x 1 ☞ n ⟼ ([n][o])

−n ⟼ ±n x i ☞ <n> ⟼ ([n][i])

Seeing the ± numbers as bipolar composites is precisely what Euler proposed:

> The square root of a given number always has a double value, thus negative as well as positive can be taken.[11]

$\sqrt{(x^2)} = \pm x$

In the intervening years this perspective has lost respect. Instead of composing the x-axis as an i to ± continuum, the newly minted +ni and −ni were assigned their own orthogonal *dimension*, creating the complex plane.

This may be asking for a significant stretch of the imagination but in a sign algebra i is a **half-reflection**

operator multiplication
i x i = −
− x − = +

half-negation
i(i) = −
−(−) = +

operator. Although multiplication treats i as an object, functionally it is an operator. i applied to i yields the negation operation. Similarly negation applied to negation yields a positive. In this operator calculus −i is a one-and-a-half negation operator. As well, ±1 can be taken as a natural way that numbers are. From the perspective of self-multiplication the negative unit, −1, is a secondary distinction, not deserving to share the x-axis with the bipolar numbers. This tension is due directly to the design decision to have − x − = +.

To offer a quick connection to James algebra, the angle-bracket — that container that is an amalgam of all conventional inverse operations — is an epiphenomenon. It can be treated as an abbreviation for something more fundamental, the **J-frame**.

J-conversion

<A> =*def*= (J [A])

When n = 1,

<o> =*def*= (J [o]) ☞ −1 = $\#^{J+0}$

[<o>] =*def*= J ☞ J = log# −1

Since self-multiplication leads to bipolarity, defining x^2, √x and i in terms of multiplication is problematic. The James forms include no concept of multiplication. Frames then permit disassociation of ±1 and ±i from their imposed definition as *products*. For example, although x^2 indicates the self-multiplication of x, the James form (([[x]][2])) does not.

Exponential Space

The exponential function uses the real numbers to indicate the extent to which a specific base is self-multiplied. Figure 38-4 shows the projection of an exponential curve f(x) = b^x onto the linear x-axis. Although x ranges over the entire number line, from negative to positive infinity, the exponential transformation converts all x values to *positive* f(x) values. Yes this situation is analogous to the obliteration of negative numbers by self-multiplication.

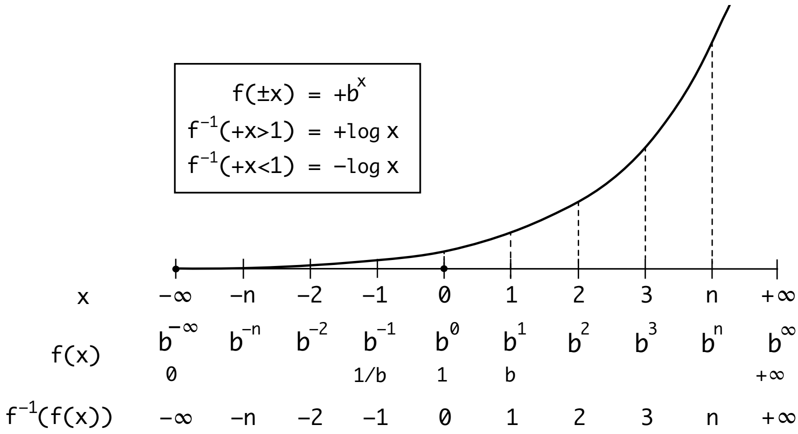


Figure 38-4: *The exponential function projected onto the* x-axis

Their resurrection will take the same path: positing a new imaginary unit indicated by a special symbol.

$$b^x = -1 \qquad \textit{therefore} \qquad x = \log_b -1 \;=\!def\!=\; J$$

The exponential function hijacks the linear x-axis values by converting them from additive entities to multiplicative entities. What was plus-1 becomes times-b, where b is an arbitrary number greater than 1, the **base**.[12] We cannot of course use times-1 cause that changes nothing. To convert additive numbers into their multiplicative counterparts, the exponential function must convert the additive identity, 0, into the multiplicative identity, 1. This conversion in effect leaves no room for an exponential value to be negative. For $f(x) = b^x$ there is literally no other side of zero. The distinction between positive and negative becomes a distinction between greater-than-1 or less-than-1 (again lending credence to the suspicion that −1 is not fundamental).

In the case of $n \longmapsto n^2$, the existent −n values on the x-axis are melded with the +n values to yield the new bipolar creature ±n. Loosely, − ⇒ ±. In the case of $x \longmapsto b^x$, the negative signs attached to x are converted into the division operation. Loosely, − ⇒ ÷. The exponential function

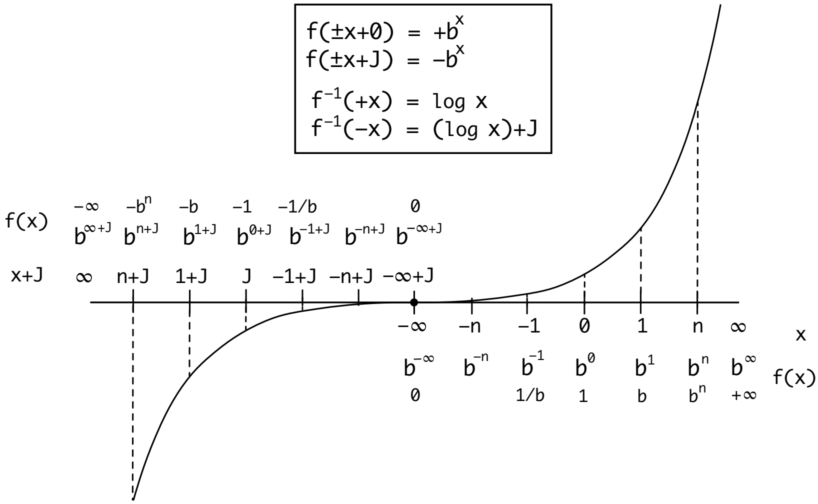


Figure 38-5: *The imaginary exponential function projected onto the* x-axis

applies to all x values, including irrationals. The explicit base form of the exponential assures compatibility by passing x through logarithmic (i.e. irrational) space.

$$(x)_\# = (([x][[\#]]))_\# \quad ☞ \quad \#^x = \#^{\#^{\log_\# x + \log\log_\# \#}}$$

What then happens on the other side of b^x? What does a negative result of the exponential function mean? As you might expect, it's imaginary and can be constructed, like self-multiplication, by creating new imaginary x values and labeling them with a new symbol.

Birth of J

Figure 38-5 shows the imaginary reclamation of negative values for the exponential function. The new imaginary domain is additive rather than multiplicative, with the x-axis calibrated as logarithms of real numbers. The negative numbers on the left side of the figure are designated by +J; they live in the imaginary ***J-land***. The inhabitants are labeled as ±n + J.

$$+n = \pm n + 0 \quad ☞ \quad n = (\ [n])$$
$$-n = \pm n + J \quad ☞ \quad <n> = (J\ [n])$$

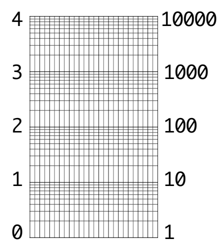
J-land differs significantly from i-land primarily because the negative numbers on the number line have not been obliterated by sign-blindness. Instead they are co-opted for use as designators of unit fractions. We then have the following relationship between the common operators of arithmetic, both direct and inverse. The diversity of arithmetic operators are indeed the same operation at different levels of nesting.

$b^{-n} = 1/b^n$

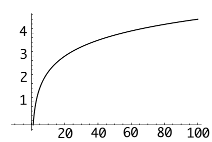
$\log_\#$ A ☞	[A]	$\log_\#$ –A ☞	J [A]
+A	([A])	–A	(J [A])
1 x A	(([[A]]))	1/A	((J [[A]]))
$\#^A$	((([[A]])))	$\#^{1/A}$	(((J [[A]])))

Logarithmic Space

The logarithmic transformation maps exponents onto the number line. Evenly spaced ticks on the number line are labeled by orders of magnitude, not by evenly spaced counting intervals. This mapping is common enough to support a special type of graph paper, **log paper**, that displays base-10 exponents as integers on the y-axis. The scaling between units is times 10, not plus 1. Log paper is used when we need to graph an exponentially large range of numbers, say from 1 to 100,000.



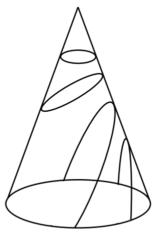
The logarithmic mapping transforms multiplication into addition by taking additive linearity as order of magnitude and forcing the linear scale to take multiplicative steps. With log paper we can add distances between coordinate values to achieve multiplication. That's also how a slide rule works. And we can see this amazing transformation directly in James forms:



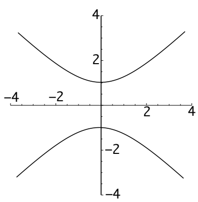
logarithm

a x b	☞	([a][b])	
log (a x b)	☞	[([a][b])]	cover
		[a][b]	clarify
		☞ log a + log b	

38.5 Conic Sections



Plane slices through a solid cone generate the conic sections: the circle, ellipse, parabola and hyperbola. These geometric curves have been studied throughout the history of mathematics, particularly by the ancient Greeks. We have explored the parabola as the self-multiplication function. The circle will be a central topic for trigonometry in Chapter 40. The conic circle, the trigonometric functions, the imaginary `i` and the logarithm are deeply connected. The conic hyperbola, the natural `base-e` and the logarithm are also all deeply connected.

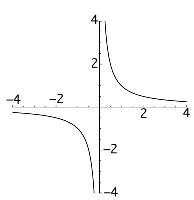


hyperbola

Here we'll briefly explore the connection between addition, multiplication, conic sections and logarithms. These relationships are so intertwined that we will take logarithms to be more fundamental to arithmetic than is multiplication itself.

The form of multiplication is `([A][B])`. A **unit hyperbola** has the algebraic form

`A x B = 1`   ☞   `([A][B]) = ( )`

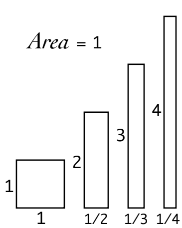


reciprocal

The hyperbola models inverse relations in which one parameter increases as the other decreases. As `A` increases along the `x-axis`, the height `B` along the `y-axis` decreases proportionally. It's precisely the same relation as between points inside a unit circle and points outside as defined by inversive geometry.

The reciprocal exhibits scaling invariance. Since the two variables are related by multiplication, this is best visualized as invariance of area.

$$f(x) = 1/x \qquad f^{-1}(x) = x$$



The void-based version of the hyperbola is

`[A][B]` = *void*

This in turn is the sum of the logarithms of the `A` and `B` coordinates of each point along the hyperbola. The logarithms of the coordinates add up to 0, turning multiplication into addition.

$$[A][B] = \textit{void} \quad ☞ \quad \log A + \log B = 0$$

With A = n *and* B = 1/n

$$\log n + \log 1/n = \log n - \log n = 0$$

The form that serves as the multiplication unit is ( ), while the unit for addition is ([ ]). This relationship is similar to the structure of the unit circle, for which

$$A^2 + B^2 = 1 \quad ☞ \quad ([A][A])([B][B]) = (\ )$$

In void-based terms

$$[([A][A])([B][B])] = \textit{void}$$

cover

The coordinates of the unit circle specify the values of the trigonometric functions, while the logarithm of the sum of their self-multiplications is 0. The coordinates of a unit hyperbola specify the values of the reciprocal function, while their logarithms add to 0.

To emphasize the family resemblance between circles and hyperbolas, we can rotate the orientation of the x-y coordinate system of the hyperbolic reciprocal function by 45°.

$$A = x + y \qquad B = x - y \qquad A = 1/B$$

$$(x + y) = 1/(x - y)$$

$$(x + y)(x - y) = 1$$

$$x^2 - y^2 = 1$$

Alternatively a hyperbola is a circle on the complex plane.

$$x^2 + (iy)^2 = 1$$

38.6 Composite Number Lines

The number line in Figure 38-6 shows our interpretation of the three *operator spaces* of James algebra,

- — **exponential:** inside round-brackets (A B)
- — **additive:** the contents of any brackets {A B}
- — **logarithmic:** inside square-brackets [A B]

exponents	$10^{-\infty}$	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{∞}
numerals	0	.001	.01	.1	1	10	100	1000	∞
logarithms	−∞	−3	−2	−1	0	1	2	3	∞

Figure 38-6: *The positive composite number line*

This composite number line shows the usually additive number line in the middle row compressed exponentially. The linear scale takes "unit" steps that multiply by base-10. The exponential scale on the top row shows these steps in exponential notation, so that $1 = 10^0$, $100 = 10^2$, etc. The logarithmic scale on the bottom row shows the logarithms of the unit steps. The logarithmic scale now resembles a conventional number line with steps of one unit each. The entire figure thus shows the logarithmic transformation as it applies to an additive number line. The exponential and the logarithmic scales accommodate the entire range of real numbers, from negative to positive infinity, leaving the center numeric scale without negative numbers. This circumstance is a consequence of the definition $10^{-N} = 1/10^N$.

The top composite number line in Figure 38-7 reinserts the negative real numbers, creating an initial problem for the exponential and logarithmic scales. The *labeling* solution is simply to add +J to the logarithmic values (and to the exponents) on the left side of zero. The logarithms of negative numbers are then their numeric magnitude *plus* J. This labeling is analogous to attaching the imaginary marker i, except that the J marker is *added* to an expression while the i marker multiplies an expression.

The lower blowup extension in Figure 38-7 shows the number line spanning from −1 to +1, completing the gap directly above it. Here we see negative values as exponents that identify smaller and smaller numbers. Rather than obliterating the negative numbers like the

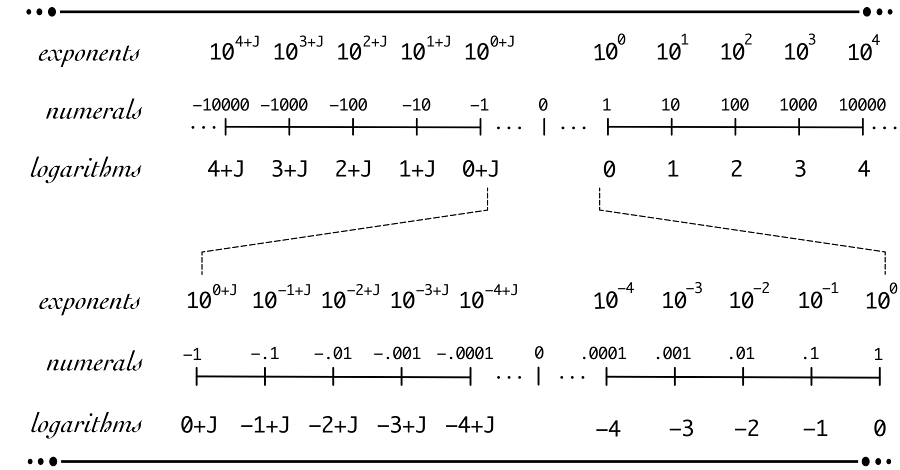


Figure 38-7: *The entire composite number line*

x^2 mapping does, here all of the negative logarithms fall between `0` and `1`. As negative exponents approach negative infinity, their associated numeric magnitudes approach `0`. On the left side of `0` the negative logarithms are all supplemented by `+J`.

Figure 38-8 serves as a summary. The reflected exponentials and logarithms on the left of `0` are *imaginary*, populating a new imaginary territory, **J-land**. But the presence of `J` is more than a label, it is how these numbers work. Notice that the exponential $10^{0+J} = -1$. The *magnitude* is `1`; the added `J` indicates only that the original value is negative rather than positive. Numbers like `−10000` have a magnitude of 10^4 and a polarity of `J`.

$$-1 \quad ☞ \quad 10^{0+J} = 10^0 \times 10^J \quad ☞ \quad 1 \times -1$$
$$-10000 \quad ☞ \quad 10^{4+J} = 10^4 \times 10^J \quad ☞ \quad 10000 \times -1$$

Consistent with J-transparency, `J` does not impact magnitude. The form of `J` is independent of its contents. `J` is also independent of the exponential and logarithmic base so that

$$J = \log_{\#} -1 \qquad \textit{and} \qquad \#^J = -1$$

J-transparency
[<(A)>] = J A

It should be clear that the exponential use of `J` *incorporates no new computational results*. Only the perspective differs.

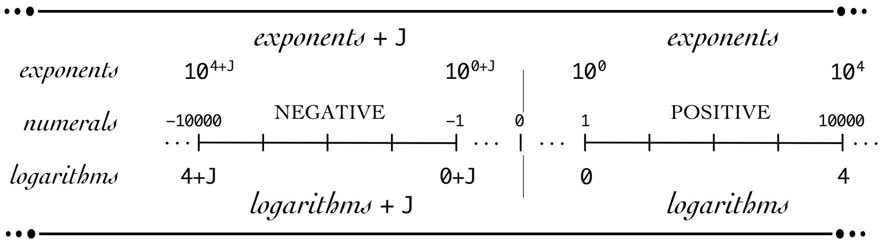


Figure 38-8: *Numeric domains on the composite number line*

Here is the general case of a negative number with an explicit base.

explicit base $\quad -B^n$ ☞ <(([[B]][n]))>

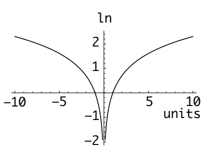
We'll immediately substitute the generic base B = (( )).

substitute $\quad$ <(([[(( ))]][n]))>$_B$
clarify $\quad$ <(n)>$_B$
J-convert $\quad$ (J n)$_B$ ☞ B^{n+J}

Adding J *to the exponent* of a negative number transfers its polarity to its exponent. Two simple examples:

-1 ☞ <( )>$_\#$ = (J)$_\#$ ☞ $\#^{0+J}$
J-convert $\quad -2^1$ ☞ <(o)>$_2$ = (J o)$_2$ ☞ 2^{1+J}

J provides a new notational method for handling mappings that themselves eliminate the negative numbers, such as $f(x) = x^2$ and $f(x) = \log x$. Within James algebra negative numbers do not need to exist on the *linear* number line, a perspective consistent with treating the angle-bracket as an abbreviation rather than a fundamental numeric form. The J notation is a *new representational paradigm* in which numbers as numeric bases are sign-free while their exponents carry both order of magnitude and polarity.



logarithms of positive and negative numbers

James Number Line

The next composite number line in Figure 38-9 shows the James forms for exponential, linear and logarithmic numbers. Every quadrant is informative. For orientation the middle row shows decimal numbers with steps marked in powers of ten. Above them are the boundary forms that correspond to these numbers. In the lower

exponents	<(oo)>	<(o)>	<( )>	<(<o>)>	<(<oo>)>	*void*	(<oo>)	(<o>)	( )	(o)	(oo)
numerals	-100	-10	-1	-.1	-.01	0	.01	.1	1	10	100
logarithms	[<(oo)>]	[<(o)>]	[<( )>]	[<(<o>)>]	[<(<oo>)>]	*void*	<oo>	<o>	*void*	o	oo

Figure 38-9: *James forms on the composite number line*

right logarithmic quadrant James digits are simply collections of units, the same as in ensemble arithmetic.[13] In the upper right quadrant the outermost container is a round-bracket which tells us that the forms can be interpreted as exponential. Below these round-bracket forms, the outermost brackets have been removed in the conversion of exponential forms to logarithmic forms.

The negative quadrant on the top left reflects the positive quadrant across *void* by enclosing each form in an angle-bracket, or alternatively by appending J. For example:

-10 ☞ <(o)> = (J o)   J-convert
-.1 ☞ <(<o>)> = (J <o>) = (J (J))   J-convert

And finally the lower left quadrant shows the James forms of the imaginary logarithms of negative numbers. For example,

$\log_{10}$ -10 ☞ [<(o)>] = J o ☞ 1 + J   J-transparent
$\log_{10}$ -.1 ☞ [<(<o>)>] = J (J) ☞ -1 + J   J-transparent

The composite number line is not a simple place.[14] We have not addressed the many types of infinity that bound these lines, nor the real numbers tucked between all possible rational numbers. Egregiously missing from the numeric number line is the James *void*, the hole filled by numeric zero. Understanding these generic forms of James numbers is a convenience but not a necessity. What is important is that once we change to expressing rational numbers as exponents and logarithms, surprises do occur. Here we found polarity in exponential space.

38.7 Equivalence of Systems

The assertion that iconic formal systems do *not* map onto symbolic systems is essential for James algebra to be of mathematical interest. If James arithmetic is equivalent to, say, Peano's arithmetic then the distinction between symbolic and iconic is inconsequential. We have just changed the appearance of things without changing any of the things themselves and without changing their behavior.

There is some trickiness even to specifying what the criterion for equivalence is. Simply put, there is a many-to-one mapping between symbolic and iconic systems. An icon is worth a thousand symbols. Just as importantly, the native concepts of a symbolic system (such as ordering, grouping, specified arity, explicit bases, negation and multiplication) do not correspond to the native concepts of an iconic system (such as void, containment, unspecified arity, arbitrary bases and nesting). We could connect the two types of representational systems via the idea of a binary `contains` relation between container and contents, however the difficulty there is that this single relation is mapped onto all the standard functions of arithmetic, with variety provided only by the varieties of boundaries. There is a severe mismatch of object types.

Homomorphism

Homomorphism is a technical term to describe two mathematical systems that are structurally the same except for changing the representation of things. Figure 38-10 shows the mechanism used to prove structural equality across different string-based mathematical systems. The algebraic definition is

$$h(a * b) = h(a) \,¤\, h(b)$$

Please excuse the ugliness. $*$ and ¤ are generic symbols for two different operations in two different worlds, *World 1* and *World 2*. The **morphism**, or *mapping function,* `h` is the roadway between these worlds.[15]

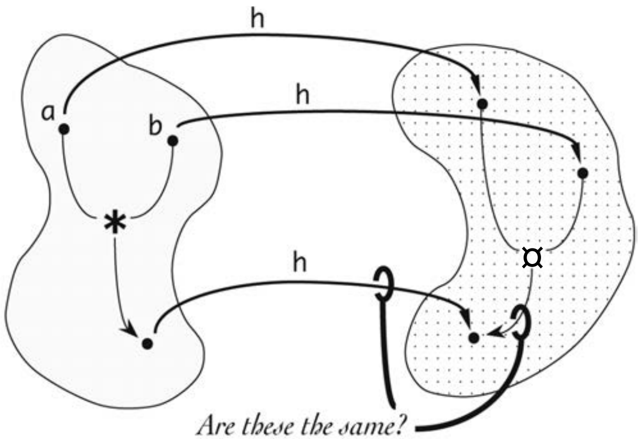


Figure 38-10: *Structure preserving maps*

The entire approach of establishing symbolic homomorphism is built upon the assumption of comparing two *symbolic* systems. The mechanism of comparison itself requires conformance to the rules of symbolic algebra. As well the mathematical concept of morphism applies only between systems with the same type of objects. Relations, for example, do not map to functions; matrices do not map to geometric curves. How then does ☞ manage to work?

The interpretation indicator, the *mapping* finger ☞, is the highway h that joins symbolic and iconic conceptualizations. The finger however does not go in two directions (which would define an isomorphism) because it *deletes* rather than maps when we shift from symbolic expression to iconic form. We can shift back from iconic form to symbolic expression by providing the missing symbolic components. In so doing we can freely create a diversity of interpretative *readings*, all of which preserve a single meaning but do not preserve the conceptual necessities of symbolic structure.

The mathematical language of mappings presumes the constraints of mathematical groups which in turn embody strictly symbolic concepts. The homomorphism picture in Figure 38-10 presumes, for example, that two objects (more generally, a fixed number of objects) will be combined by the `*` and `¤` operators. Containers, in contrast, contain any cardinality of forms while pattern transformations can be composed of zero, one or many concurrent but separate pattern matches.

Leave Void Out of It

The trouble begins when `h` goes off looking for `0` in the iconic world. `h(0)` and `h(0+0)` *do not exist*. And what is `¤` in the iconic world? The only potential operation we have is the relation of containment. Certainly *void* does not contain *void*. Neither does `¤` exist because there is no iconic operation capable of acting upon *void*, much less finding something that is not there. Although we can safely say from the symbolic perspective that *void void* = *void* this construction violates both structural and interpretative intentions of void-based systems. The only available operation is that one container `contains` another, given that both exist. *Representational existence* is never a question in a void-based calculus since void-equivalent forms can be brought into existence at any time. *Semantic existence*, on the other hand, is not supported within a pattern-matching system. It is also illegitimate in a mathematical system to convert a meta-token in the metalanguage such as *void* into an actual object within the mathematical system. A deeper problem is that all void-equivalent forms pervade all depths of any manifest form and can be arbitrary brought into representational existence within any container and with any number of replicas.

Pattern Matching

The complexities of maintaining a mapping can be redefined by a variety of implementation strategies, something mathematicians do not usually consider. Pattern-matching, for example, does not include the concept of a function. What would pass for a function with regard to output could include pattern algorithms and constraints that violate the assumptions of functional algebra. Differences at the foundational level are not so easily set aside as mere implementation details. It is the transitions inside each domain that do not match up, even though the final results do correspond. The mechanisms on the symbolic side must take into account commutativity and associativity of operators, while the iconic side lacks these concepts. The iconic side can proceed with arbitrary bases (using #) while the symbolic side cannot. The *implementation details* within each system are vastly different, so much so that the internal diagrams in Figure 38-10 cannot be aligned. The inability to be able to map the *theory of functions* onto an iconic pattern-matching regime can be severe. James algebra for instance has no concept of either multiplication or division, although specific compound patterns can be interpreted as multiplication and division. Dominion takes an excursion through an infinite form, as another example, in the process of "multiplying by zero". The pattern-matching mechanism, although assumed by *both* systems, is vastly different in each. **Functional pattern-matching** proceeds step-at-a-time while respecting the definitions of each specific function. **Iconic pattern-matching** is in parallel with no functional or relational boundaries. Patterns are determined solely by matching boundary *structure* without specific functional signatures.[16]

Figure 26-1 in Volume II lists how the essential concepts of symbolic algebra are deconstructed by an iconic system. In this section we have yet to consider subtraction or division or logarithms but we know that the symbolic side gets quite messy while the iconic side requires one additional constant, J.

The Finger

To make the finger work across symbolic and iconic domains, we have ignored the details of formal mappings, making only side comments about

- the hole associated with the concept of `0`
- the replacement of value by form
- the unlimited void-equivalent forms in the abyss that themselves have no meaning
- the vast differences between symbolic functions (and relations and sets) and iconic structural pattern-matching, and
- the many-to-one mappings that permit multiple readings and multiple valued forms.

Going from James iconic form to symbolic expression, the finger works because it is coming from a tiny world that includes only configurations of two types of symbolic structure (exponential and logarithmic interpretations of `( )` and `[ ]` respectively) and one constant, `J`. Although this mapping generates quite exotic symbolic expressions, the conceptual structure of well-behaved symbolic arithmetic is highly redundant and is, underneath, precisely that small.

Going from symbolic expression to iconic form, the finger works because the iconic form is tolerant of the symbolic redundancies, needing only three pattern transformations to remove the illusory diversity. The vast majority of the clean up is the application of `clarify` to strip away layered complexity hidden in the symbolic distinction of most functions. Functions are excellent *macros* for algebraic simplification and transformation but are at best only that, tools for conceptual organization rather than stand-alone unique concepts. Put another way, the finger does not work in purely symbolic terms, it works because the iconic approach lets us avoid the complexity of symbolic morphisms.[17]

38.8 Remarks

There are several important aspects of the mappings in this chapter.

- The interpretation of forms consisting of round- and square-brackets ranges simultaneously across conventional exponential, linear and logarithmic scales.
- Translation and rotation can each be expressed as a composition of two reflections, providing a natural geometric interpretation of the angle-bracket, which in turn is the constant J.
- Logarithms and exponents re-purpose negative numbers, providing room on the composite number line for the new domain of J.

The ideas of mapping and morphism are themselves not well-suited for void-based pattern transformation systems. What we have is *faux amigos,* false friends expressed in clashing languages. What is surprising is that we can cross the finger of interpretation at all. The exposition in these volumes is deeply biased toward current technique with only occasional partial excursions into the vastly different realm of iconic thinking. We will return to the formal iconic image languages of Volume I in Chapter 44.

In constructing a map between conventional and boundary forms we have been picking low fruit by assuming that the boundary forms that correspond to natural operations of conventional arithmetic are somehow special. We have concentrated on the form of multiplication ([.][.]) while treating forms such as ((.)[.]) and ((.)(.)) and [(.)(.)] lightly. We have also introduced a fairly radical shift in notation: the polarity property has been moved into exponential space, leaving magnitudes as pure signless forms. Euler's equation moves the polar coordinate system of the complex plane into exponential space in order to explain the behavior of J. In the next chapter we'll unite the J-frame with Euler's concept of J to simplify both his explanation of rotation in the complex plane and his resultant logarithms with an infinity of values.

Endnotes

1. **opening quote:** E. Tufte (1983) *The Visual Display of Quantitative Information* p.180.

2. **every point on the circumference is zero:** And in three dimensions, Tristan Needham observes

> The close connection between inversion in a circle and reflection in a line also persists: reflection in a plane is a limiting case of inversion in a sphere.

T. Needham (1997) *Visual Complex Analysis* p.135.

3. **three types of reflection, or inversion, are fundamentally different:** Reflection is a more general concept than reflection across a line or across a circle. Given some restrictions imposed by the concept of continuity, reflection through an arbitrary function is well-defined. James Reflection is an inversion through `J`, as specified by the J-occlusion theorem.

`A (J [A]) =` *void*

4. **curly braces to represent a pair of coordinates:** Curly braces are also used for sets, but not here. These coordinate curly braces also should not be confused with the use of the *generic James boundary* in the prior chapter.

5. **interact with the plane and its coordinate system externally and "objectively":** Viewing a surface from the third dimension allows us to identify the *curvature* of the surface.

6. **angle of rotation to the (extrinsic) `x` and `y` distances:** The `tangent` is the ratio of `sine` to `cosine`. `Arctan` is the inverse of the tangent. Whereas the `tangent` gives us the ratio if we know the angle, `arctan` gives the angle if we know the ratio between the `x` and `y` distances.

`arctan (3/4) ⇒ 37°`

7. **as well as all other void-equivalent forms:** Symbolic `0` continues to be a problem for iconic systems, while *void* is a problem for symbolic systems.

8. **reflecting the natural numbers as unit reciprocals:** Seeing a fraction as a circular reflection is diligently ignored in contemporary math education, possibly because of the relatively new and emotionally turbulent discovery of non-Euclidean geometries.

9. **both on the same line and both without limit:** Well after the number line has made its impression, we might learn that there is a *larger* infinity of real numbers stuffed between the fractions. Later still we may be told that the uncountably dense number line is insufficient to hold *all* the numbers, we will need another dimension no less to explore the structure of complex numbers on the complex plane.

10. **convert a generic quadratic function `f` into the simpler squaring function `F`:** The generic form of the quadratic in the example is the **vertex** form, in contrast to the more familiar **standard** form, $F(x) = ax^2 + bx + c$. The illustrated transformations are the **similarity** group (*translate, rotate, scale*).

11. **thus negative as well as positive can be taken:** L. Euler (1771) *Complete Guide to Algebra* §150.

A reason we condense rules that provide *options*, a reason we use exclusive or, `XOR`, rather than inclusive or, `OR`, is that our mathematical and cognitive histories have had a great difficulty with the logical concept of disjunction. Having a number with *both* polarities is not convenient for functional thinking. When traveling a path of transformation, an `OR` branch offers a *choice* of the next transformation. Using `XOR` logic to choose one path is *sequential processing*. Computationally we can go back to try the other path if we have made a poor choice. Making a wrong guess is expensive, guaranteeing that the computation becomes intractable since the number of choices can increase exponentially with no assurance of the concrete termination that is enforced within physical reality. The *parallel processing* option is `OR` logic; take all available paths at the same time. Technically, we are exchanging SPACE in the form of multiple concurrent processors for TIME in the form of sequential steps and backtracking. The `XOR` logic of one-or-the-other, in both concept and process, reduces or at least delays complexity. The `OR` logic of one-and-both maintains *ambiguity*, which has been inappropriately interpreted as failure to make a decision.

12. **where b is an arbitrary number greater than 1, the base:** A unit fraction base reflects the exponential function through the vertical line $x = 1$ since

$$b^{-x} = (1/b)^x$$

13. **collections of units, the same as in ensemble arithmetic:** Ensemble arithmetic is described in Chapter 2.2 of Volume I.

14. **The composite number line is not a simple place:** I've attempted to condense a great amount of structural information into small diagrams and hope that the diagrams are not impenetrable.

15. ***mapping function,*** **h is the roadway between these worlds:** For example consider two sandwich shops, SandOne and SandTwo. You go to SandOne and buy a ham sandwich with extra mustard. Now you go to SandTwo, find their ham sandwich and ask for their extra mustard. If you end up with the same ham sandwich from both shops, then there is a morphism between the ham sandwiches. The concept is much broader, the sameness might include all sandwiches and all extra condiments. And, of course, typical of math, we forget all the actual details. The sameness of sandwiches does not include the other factors that make life interesting (type of bread, quality of mustard, cost of sandwich, pleasantness of service,...).

16. **boundary structure without specific functional signatures:** In computer science a *signature* specifies the symbols used to identify a specific function, the arity and ordering of the function arguments, the types of each input, output and variable, and often incidental information such as polymorphisms that identify symbols that may be used across different types. Boundary structure is usually insensitive to arity, ordering, variable types and other non-boundary structural constraints.

17. **the iconic approach lets us avoid the complexity of symbolic morphisms:** I am certain that professional mathematicians look *through* the symbolic haze into the heart of mathematical beauty. In coming from a different field (silicon programming and design) that insists upon extreme minute rigor while also being responsible for the education of novices, the narrative in these volumes addresses more the pragmatism of symbolic math than the beauty of mathematical abstraction.

Chapter 45

Return

Math was always my bad subject.
I couldn't convince my teachers
that many of my answers were meant ironically.[1]
— Calvin Trillin (1988)

Volume I presents two iconic approaches to the representation of the formal structure of arithmetic. It also serves as an introduction to a different way of thinking about formality. Volume II contrasts postsymbolic arithmetic to the established symbolic foundations of arithmetic developed well over a century ago. This volume explores the iconic constant J and applies void-based thinking to several areas of elementary mathematics.

45.1 Evolution

At the turn of the twentieth century the mathematical community adopted a radical plan to put mathematics on a firm foundation. The idea was **symbolic formalization**, the representation of concepts using encoded symbols that bear no resemblance to what they mean. Also at the turn of the twentieth century the American logician C.S. Peirce contrasted *symbolic, indexical* and *iconic* notations within his development of the field of **semiotics**. Peirce was the first to formalize iconic logic with his **Existential**

iconic forms look like what they mean

Graphs, a form of logic based on containment within two dimensional spatial boundaries.[2] Peirce believed that his graphs *illustrated* the process of logical thinking. In 1967 G. Spencer Brown reintroduced Peirce's iconic system in a more general algebraic style as *Laws of Form*.

Return to Postsymbolism

Today mathematics remains symbolic while other communication media have evolved into visual and interactive experiences. We write on lines of paper but what of the communication of artists and sculptors and film-makers? We speak in a linear flow of words, but what of the non-linear flow of music and poetry and dance? We read books that display strings of tokens but what of the display of illustrations and photographs and videos and websites? Our digital computation tools can render both lines of text and dynamic images but that which is encoded within huge strings of binary digits has no knowledge of context or environment. Symbolic computation cannot make a distinction, cannot have an idea; it can neither know nor deceive itself.[3]

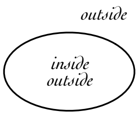
Although symbolic mathematics may be eternal, the context, meaning, relevance, interpretation and worthiness of any aspect of mathematical symbolism is involuted by physical time and by cultural change. **Belief in mathematics** is *not* eternal. Indeed a theme of postsymbolism is that formal decisions and commitments made in one era can in a later era become dubious. Hilbert sought to design a mathematics that supported entirely its own validity, without reference to the context that contains it. Prior to our modern understanding of cybernetics, ecology and embodiment, Hilbert's dream of metamathematics was a great improvement that literally laid the groundwork for the computational age. But by design metamathematics cannot be *relevant*. Symbols are necessarily obscured, obliterated, by their meaning. Formal symbol systems are designed to deceive. There is no there there.[4]

Digital technology has spurred an evolution in the representation and acquisition of mathematical knowledge. Mathematics is shifting to visual rather than textual forms of expression. Venn diagrams, Feynman diagrams, diagrammatic reasoning, cellular automata, fractals, knot theory, string theory, category theory, silicon circuit design; each of these fields relies upon the iconic representation of formal systems. Like geometry these fields can be seen as addressing inherently multidimensional concepts. Formalism is escaping its symbolic constraints.

But we face a severe limitation: *formalism itself is currently defined by symbolic strings*. Created in an era without film and television and video gaming and telephones-that-are-cameras and social media, symbolic formalism is permeated with accidental structure imposed by a medium of expression that obscures the content being expressed. Via the syntax/semantics barrier, symbolic form is intentionally blind. It cannot access its own meaning. It lacks a direct facility for feedback and self-modification. Attempting to isolate any aspect of reality or fantasy from the context/environment that provides its sustenance and vitality is delusional behavior. *Thought is a system*, integrated with its mental carrier which is itself integrated with its physical carrier which is itself integrated with its experiential context which is itself inseparable from the warp and weft of reality. Today it is both dangerous and destructive to believe in a Platonic reality separate from the actual reality since existential questions stretch well beyond the isolated security of formal and philosophical thought and well outside of the contextual disassociation of silicon computation.

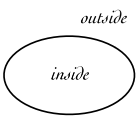
45.2 Crossing the Boundary

After working with *Laws of Form* for over a decade, I learned to recognize the depths to which Spencer Brown had deconstructed our notions of Truth and rationality. In my mind, axiomatic Crossing and Calling boiled over from logic into the neurophysiology of thought and into the fundamental philosophies of Western culture. Taking *void* seriously forces one to look at the unity of life rather than escaping into the ease of dualism. It is no longer appropriate to pretend that our thoughts can be separate from our actions. In *boundary logic* this is particularly clear since logical boundaries are **semipermeable**, the outside has complete access to the inside. I explored a void-based perspective for two more decades, applying semipermeability to the most challenging logical tasks available: programming languages that support provably correct code and the optimization of million gate silicon circuitry. Iconic *arithmetic* grew from the rigorously computational foundation of iconic *logic*. Learning how to interpret Spencer Brown's work involves connecting abstraction to experience.



logic

Logic Begets Numerics



numeric

An impassible chasm became obvious. Aside from a few professionals, folks don't really understand or have much interest in *logic*. People are extensively irrational.[5] We all however must face numeric arithmetic from an early age onward. It seemed like a good idea to construct *iconic arithmetic* as a basis for communicating the importance of iconic thought to cybernetic philosophy.

Which shared beliefs are most universal and most entrenched across the cultures of the world? What would it be like to deconstruct and then to reconstruct the skills taught to every school child? Mathematics and language are the universal basis for modern primary education. In schools logic never shows up. Language skills are of course fragmented across the many human languages, but the arithmetic of whole numbers is relentlessly present in every child's schooling. What fascinates me is the exotic challenge of exploring an *alternative* to one of the few things that nearly every culture considers to be a vital necessity.[6] This effort is particularly appropriate in today's age of the **tyranny of numbers**.[7] Our institutions and businesses are becoming increasingly reliant on AI pattern-recognition systems combined with unimaginable amounts of digital data to make decisions about both the strategies of organizations and the fate of individuals.

The task then was set: To demonstrate that *what we take as universal is simply a design decision* that we have chosen from a diversity of alternatives. The motivation is grounded in a belief that humankind is not at its apex of sophistication. What we believe today, I believe, will look as antiquated in the coming century as the pre-computational construction of formal mathematics a century ago looks today. For proof of principle what is needed is a clear, equally capable alternative that provides an entirely different way to think about arithmetic, an alternative that is *not isomorphic* but rather that is constructed on a completely different ground. Spencer Brown has provided that ground at the absolute minimum: *void* rather than zero. The accumulating round-boundary is an easy analog for tally arithmetic. Kauffman arithmetic provides an iconic resolution to the grouping problem encountered by every civilization that has accumulated too many individual tally marks. Addition and multiplication of numbers is thus addressed by tally arithmetic augmented by substitution.

accumulate
( ) ( ) ≠ ( )

group base-2
o o = (o)

merge
(A)(B) = (A B)

Spencer Brown's tectonic shift in the structure of formal thinking suggests how to extend tally arithmetic into rational and real numbers. The innovation is to construct a second kind of boundary unit that is non-accumulating (Unify) and that cancels the round unit (void Inversion). The Law of Calling is a model for a non-accumulating distinction. The Law of Crossing provides a model of how to compose distinctions with their inverses. Refocusing on *containment as the only relation* eliminates the numeric processes of **counting** and **functions**. The singular relation removes addition and multiplication from James patterns, placing these numeric functions

unify
[ ] [ ] = [ ]

void inversion
([ ]) = [( )] = *void*

call
⟨ ⟩ ⟨ ⟩ = ⟨ ⟩

cross
⟨ ⟨ ⟩ ⟩ = *void*

instead into the interpretation. The James calculus then stands solely on *containment patterns* rather than on our familiar arithmetic constants and operations. The operations of iconic arithmetic consist of putting parts into mutually inverse containers, `([.])`. When we add we put the part inside the dual container, `([a b c])`. When we multiply we put each part into a container and then put those together into the inverse container, `([a][b][c])`. Divide and subtract require that we toss the constant `J` into an appropriate location within the containment pattern. Computation proceeds by *pattern-recognition and rule-based substitution* that implements replacement subforms by axiomatically equivalent alternatives.

45.3 Violation of Symbolic Canons

Iconic Arithmetic is an attempt to meet mathematics on its own ground, not at the psychological underpinnings of philosophy but at the formal rigor of axiomatic structure. The enterprise is constructed on several Principles that guide and determine the structural constraints on numeric iconic form.

void =

Principle of Void: *Void* has no properties.

`( )` ≠ *void*

Principle of Existence: Something is not nothing.

`( ) = ( )`

Principle of Identity: A distinction distinguishes itself.

`(a b)`

Principle of Containment: There is the only relation, containment.

`[a b] = [a][b]`

Principle of Independence: All forms are mutually independent.

`a` = *void* ⇒ *void*

Principle of Void-equivalence: Void-equivalent forms are syntactically inert and semantically meaningless.

One other Principle constrains the alien numeric creature `J`.

`[a J]` ⇏ `[b]`

Principle of Incommensurability: `J` does not interact with other numeric forms.

In a final affront to arithmetic, the empty square-bracket is even stranger than `J`, `[ ]` is not numeric. The non-accumulating square unit is guided by two Principles that isolate numeric form from the erosive tendencies of its antithesis.

Principle of Non-numeric Form: After reduction, non-numeric forms include a square unit.

`(a [ ])` $\nRightarrow$ `(b)`

Principle of Indeterminacy: Any numeric form in the same container as a square unit is indeterminate.

`(a [ ])` $\Rightarrow$ `?a?`

Needless to say, there is very little in the above Principles that reminds one of the foundations of symbolic arithmetic.

Symbolic Dogma

It's not surprising that the structure and transformation of postsymbolic form immediately violates several canons of the symbolic dogma.

- *Meaning is stored and manipulated in strings of symbols.*

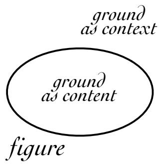
Pattern-matching exposes the *accidental structure* manifest in the representation of concepts by token strings. The symbolic doctrine casts strings as the static data structure within which control algorithms enact transformations. It is posited that knowledge can be gleaned by rearranging the linear ordering of tokens. Iconic containment, nesting itself, erodes the symbolic canon of separation of object and process, of data from algorithm. James forms are purely *patterns of containment*. Containment embodies parallelism, distributed processes, transformation *across* nested forms, and independence of every form within every container. Meaning is perceptual, interactive, experiential.

`(a b c)` $\Rightarrow$ `(b c a)`

foo
baz *bar*
$\Rightarrow$
foo *baz*
bar

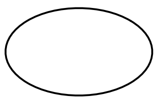
- *Image and experience do not support formalism.*

The 19th century trauma of non-Euclidean geometry grew into a ban on embedding formal meaning into diagrams and pictures. Careful structural definition can of course overcome this overt bias. A common difficulty with pictorial formalism is the distinction between figure



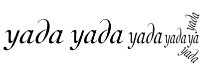
and ground. A container *cleaves* the ground and *is* the figure. Containment not only eliminates the figure/ground duality, it also allows both figure and ground to have mutually defined meaning as content and context.[8] Containers both separate and connect.

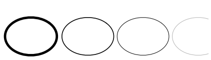
- *All concepts must have a representation.*



String languages treat the “blank space” as just another type of token, one that is given a specific meaning as a separator of strings that represent concepts. Concept, to exist, must be expressed by an arranged sequence of encoded tokens. Containers in contrast provide an internal emptiness that is not enlisted to distinguish concept from structure. The container boundary itself provides conceptual distinction. Emptiness, a contextual *void*, is then free to be employed as a structural tool. Concept can be implicit. Structure can be void-equivalent.

- *Replication is free.*





Symbolic replication is the source of symbolic complexity. Rearrangement is the primary mechanism for decoding concepts that are tangled and obscured within an arrangement of symbolic replicas. When the sequence of rearrangement steps is ambiguous, the art of finding meaning explodes combinatorially. Iconic display in contrast is spacious enough to support unique forms without replication. Illusory forms are *pervasive* throughout an iconic structure. These forms can be freely deleted and *freely constructed* anywhere while inducing minimal structural complexity. During computation void-equivalent forms can act as catalysts for reduction of significant form.[9] Only irrelevant replication is free. Irrelevant replicas are *motivated* only as ephemerals and once identified can be discarded without loss.

- *Process is rearrangement, not deletion, of structure.*

Strings collect and store structure that might be information. The pernicious NOT of logic allows storage of contradictory information. Structure is converted into information via accumulation, via the construction and retention of more structure. Strings cannot self-prune via occlusion, instead they encourage redundancy. Information is positively associated with quantity. Massive redundancy has been discovered within string representations, due not only to accidental structure supported by the linear medium but also due to irrelevant structure embedded within the string representation of concepts and operations.

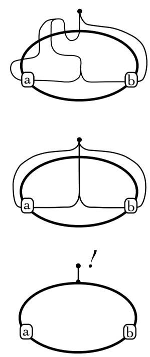
this

this
or that

this
or that
and not this

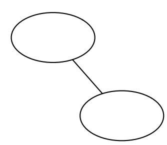
this
or that
and not this
therefore that
!

Spencer Brown's startling innovation of defining specific forms to be void-equivalent permits iconic structure to be identified as irrelevant. Iconic transformation increases information by deleting structure. Deletion increases the density and quality of information. This new perspective has evolved into **constraint-based reasoning**. *Eliminate* what is NOT to arrive at what is possible. Rather than considering a solution, consider all solutions by omitting the non-solutions. Computation moves forward by deletion of contextually irrelevant and redundant form. The structural independence of all forms makes rearrangement in space unnecessary.[10]

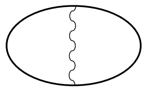


- *Meaning requires dualism.*

The void-based structure of Peirce's and Spencer Brown's iconic logics challenge a foundational assumption of Western thought, that rationality requires dualism. Duality is subtly embedded within the arithmetic and algebra of numbers as inverse operations, as the self-similar structure of repeated addition and multiplication, as polarity of value, and as the object/process distinction. Duality is embedded in string notation as group theoretic structure (commutativity, associativity, arity), as absence of parallelism in both expressions and processes, as the semantics/syntax barrier, as forced representation of both relevant and irrelevant expressions, and as the exclusion of the active Agent (the mathematician) from the notation.



duality



unity

Dualism dominates logic as the concepts TRUE and FALSE and as the dual perspectives of AND and OR. In contrast, the *logic* boundary $\langle\ \rangle$ can be interpreted as TRUE, in which case FALSE is the *void* within TRUE. Since FALSE is void-equivalent in iconic logic its non-existence makes *Laws of Form* a logic of *unity rather than duality*. Truth is confounded with existence.[11] A distinction, as represented by a single boundary, accommodates both independence and unity. Context, concept and content are a singular system.

45.4 Non-conformity

There have been several occasions during this exploration when the James calculus has lead to results that do not conform with conventional mathematics. The easy observation is that non-conforming results may require further work to understand, or they may be errors, or inappropriate, or excessively deviant, or simply wrong, or not even wrong.[12] There are certainly circumstances in which I have attempted to push James calculus to the limits of my understanding. It should not be surprising that some anomalies occur, particularly in places where conventional mathematics behaves anomalously within itself. There are a very few areas in which the James calculus appears to be self-contradictory, most of these have been addressed by imposition of rather arbitrary constraints to eliminate destructive forms. The occasions in which James forms substantively deviate from a consistent interpretation are indeed rare, given the very substantive differences between iconic and symbolic technique. There are a few occasions in which poor computational technique embedded within conventional mathematics is a source of problematic behavior. And there are a few occasions in which the mapping to exponential and logarithmic expressions appears to be presumptive. In these cases the James formal structure stands out as underneath and independent of conventional arithmetic. There are of course large tracts of conventional mathematics that I've elected not to address. The guiding theme has been to limit exploration to elementary, high school and first year college mathematics for non-STEM students, with the obvious exception of innovations such as J and some applications to calculus derivatives, complex numbers and infinity. Here is a brief overview of the innovative and the non-conforming results in this volume.

General Computational Technique

Mathematical systems use a wide variety of computational methods (deductive inference, induction, recursion, iteration, substitution of various types, infinite series, mapping, etc.). Almost all eventually reduce to rule-based transformation of formal textual structure, what might be called *algebraic technique*. Iconic form supports a much wider diversity of transformational strategies including parallelism, structure sharing and elimination of accidental structure. Iconic pattern-matching is very challenging to implement within digital software while maintaining the purity of iconic principles. Software is of course string-based, however the networks of transistors that comprise digital hardware are inherently parallel, spatial and void-based. Software embraces the textual duality of 0 *and* 1; hardware operates by the *presence or absence* of an electrical signal along a wire.

Behavior of J *(Chapters 34-36)*

Figure 34-1 lists the features of this resurrected imaginary value while Figure 34-6 lists some of the worrisome properties of J.

Tally failure *(Chapters 34.4, 35.3 and 44.6):* The equation x + x ≠ 2x violates the Distributive property that connects addition to multiplication. However tally failure is a natural consequence of the J-void theorem. An intervention is to recognize that J does not accumulate even though it is numeric. Suppression of accumulation is quite

J-void
J J = *void*

natural when one considers the extra-logical attachment of a counting mechanism required for iterative, inductive, oscillating and circular functions. *Counting repetitions requires explicit mechanism*. This weakness is attributable to hidden assumptions in conventional technique.

`a + J ⇏ b`

Incommensurable *(Chapter 35.3)*: That `J` does not add to natural numbers is a property already pioneered by `i`.

Sign-blind *(Chapter 35.3):* The bipolar property is shared by `i`, `square root` and many other conventional functions. In order to maintain a working concept of equality, J-self-inverse cannot be used for local replacement of equals by equals. All `J` forms (explicit and implicit) must be replaced globally and concurrently within an equation. Euler's intervention is better: recognize the prevalence of bipolar and multivalued functions within the laws of algebra.

J-self-inverse
`J = <J>`

Arrangement inconsistency *(Chapter 35.3 and 36.6):* **Loss of confluence** is a serious issue associated with tally failure and is the reason for suppressing accumulation of `J` tokens. The constraint not to `collect` or `disperse` replicas of `J` is necessary to avoid contradictory conclusions. J-fractions also rely on a non-accumulating `J`.

`([J][2]<[2]>)`
`⇏ ([J J]<[2]>)`

`J` ***as a polarity exponent*** *(Chapter 38.6):* The computational use of `J` is unique, particularly the creation of sign-free numbers by moving polarity information into the exponential space as the presence or absence of `J`.

$a^{k+J} \Rightarrow -a^{k}$

`J` ***as a more elementary imaginary*** *(Chapter 39):* The imaginary constant `J` is additive and structurally simpler than the multiplicative `i`.

`i = (J/2)`

Base-free Differentiation *(Chapter 37)*

Differentiation of base-free forms suppresses the logarithmic conversion constant that occurs when changing from one base to another. There is no equivalent conventional concept.

Rational Reflection *(Chapters 39-40)*

Fractional reflection *(Chapters 39.7 and 40):* Reflection along a single axis suppresses the complex plane by quantizing irrational rotation into rational fractions.

Fractional polarity *(Chapter 40.6):* Positive and negative polarity, as well as the complex plane, are special cases of arbitrary rational fractions of polarity.

Reflective trigonometry *(Chapter 40.6):* Rational reflection without polarity changes a few trigonometric identities. Euler's equation and the double-angle and half-angle formulas are reduced by a factor of 2. The Pythagorean theorem and the tangent function collapse.

Infinity *(Chapters 41-43)*

Numeric and non-numeric forms are intimately integrated within the James notation as *iconic polymorphism* of object and operator, removing the type confusion of infinite expressions.

numeric unit ( )
non-numeric unit [ ]

The form of infinity *(Chapter 41.3 and 41.4):* The representation of negative infinity as [ ] and positive infinity as <[ ]> is a reversal of the interpretation of the angle-bracket that points to an unusual relationship between infinity and polarity.

Absorption of polarity and magnitude *(Chapter 35.3):* The choice and sequencing of valid transformation must be managed with structural restrictions so that the square unit does not absorb magnitude and polarity information. Since positive infinity and negative infinity are deemed in all cases to be different, absorption of polarity information must be managed closely.

Infinite Interpretation axiom *(Chapter 41):* Mixing an external interpretation of a formal system with the internal axioms of that system has substantive negative effects on interpretability and application. The mitigation is to restrict Infinite Interpretation to forms with rare applicability that have no other avenue for reduction.

Complex infinity *(Chapter 41.7 and 41.8):* The double-square unit `[[ ]]` carries complex infinity into James forms with dubious results. A central issue is whether or not an infinite unit can or should `absorb` polarity, or in the James case `absorb J`. The triple-square unit might be inaccessible to consistent computation.

Reflective infinities *(Chapter 41.8):* The dimensionality of a complex infinity with both magnitude and direction are exacerbated by James reflective infinities that support a different imaginary axis for each of the `N` unit fractions of `J/N`.

Indeterminate forms *(Chapter 42):* The easily identified single indeterminate form `[ ]<[ ]>` provides an innovative tool for organizing indeterminate forms, although it does come with further restrictions to avoid sign-blindness and inappropriate absorption.

Non-conforming exotics *(Chapter 43):* James exotic forms sometimes reduce to non-standard results. Infinite summation of zero yields an indeterminate conventional expression but a James *void*. It is more difficult to ignore the indeterminism of 1^∞ and 0^0 in James form. Several conventional results yield a complex infinity while James forms result in specific forms of infinity or indeterminism. The polarity of $-\infty$ needs to be specifically factored into $(-1 \times \infty)$ for James reduction to yield consistent results. `J` exacerbates the conventional logarithmic collapse of polarity at infinity.

45.5 Grand Strategy

Daniel Kahneman
1934-
Nobel prize 2002

Introducing the innovative world of iconic formal thinking is bound to be extremely challenging. Daniel Kahneman observes, “People tend to assess the relative importance of issues by the ease with which they are retrieved from memory."[13] Not only is there virtually no memory of nor experience with iconic formalism, almost all memory associated with arithmetic is saturated by experience with symbolic expressions. Further, our interpretation of James patterns lands in unfamiliar territories within symbolic arithmetic.

There seems to be little motivation for conceptual justification of an *exploration*. Instead these volumes launch us into an experiential postsymbolic territory where interactive structure condenses symbolic complexity, where form and meaning are united, where objects merge with processes, where absence is a primary conceptual tool, and where the viewing perspective of the reader is directly implicated within the form.

There is a fairly wide awareness of the integrative beliefs driving postsymbolism: Haeckel's ecological holism, Peirce's iconic rationality, Spencer Brown's void-based axioms, von Bertalanffy's general systems, Bateson's cybernetic pattern-that-connects, Lovelock's Gaia, Prigogine's self-organizing systems, Tufte's visual information, Varela's embodied cognition, Wolfram's new kind of science, Buddhist and Vedic metaphysics.[14] Although arithmetic is often enlisted to provide examples that support holism, none reach in to *define arithmetic as a form of experience*. In contrast to Hilbert's metamathematical formalism built solely on isolated abstraction, postsymbolism as a philosophical perspective lies beneath the surface of iconic form as concrete experience. The postsymbolic alternative holds within it an implicit critique of both Western dualism and objective rationality. Dualism is the mechanism, the *excuse* so to speak, that allows us to indulge in the fantasy of an objective reality external to our experiential context. Radical reconstruction of cherished formal beliefs incorporates a broader commentary on the conceptual underpinnings of rational thought in particular and of epistemology in general. *Symbolic representation does not lead to embodied knowledge.*

45.6 Remarks

Necessity
— the author (1980)

Like an empty mirror reflecting an empty room
With nothing around and nothing within
The blank pages of an unwritten book,
The fountain pen still full of ink,
The author without thoughts
Fills the empty page
Expressing without necessity
Nothing at all.

Endnotes

1. **Opening quote:** C. Trillin (1988/2012) *Quite Enough of Calvin Trillin* p.4.

2. **logic based on containment within two dimensional spatial boundaries:** C. S. Peirce (1909) MS 514 Existential graphs. Online at `http://www.jfsowa.com/peirce/ms514.htm`

3. **it can neither know nor deceive itself:** The conceptual limits of computation are illustrated by many self-referential questions that cannot be addressed by a computation. The famous **halting problem** asks a computational process: *When will this process be done?* If a computation is able to know when it finishes, it must have sufficient information to actually complete the computation, and therefore can use the information of when it finishes to reach the finish *faster* than is indicated by its answer. There are **modal logics** that provide operators with enticing names such as *undecided* and *know*. Similarly I can write words like *I know that I do not know*. Neither qualify, of course, as self-knowledge.

4. **designed to deceive. There is no there there**: More accurately what remains is metaphysics, an appeal to a semantic magic that connects symbolic structure to experience. We provide that magic by disembodying cognition.

5. **are extensively irrational:** D. Kahneman (2011) *Thinking, Fast and Slow*.

6. **nearly every culture considers to be a vital necessity:** This perspective was somewhat inspired by Stephen Wolfram's book *A New Kind of Science* (2002). Wolfram addresses the universal belief that the physical sciences are fundamentally united with symbolic mathematics by providing an alternative description of Science based on the iconic form of cellular automata.

7. **appropriate in the age of the tyranny of numbers:** "So number will now manifest itself, without limit, as tyranny." A. Badiou (1990) *Number and Numbers* § 6.14. Here Alain Badiou is referring to the abstract structure of our number system. More political and sociological examples include:

N. Eberstadt (1995) *The Tyranny of Numbers: Mismeasurement and misrule*.

D. Boyle (2001) *The Tyranny of Numbers: Why counting can't make us happy*.

J. Williams (2016) *Quantifying Measurement: The tyranny of numbers*.

S. Ball (2017) *Governing by Numbers: Education, governance, and the tyranny of numbers*.

J. Muller (2018) *The Tyranny of Metrics*.

8. **to have mutually defined meaning as content and context:** The page provides a ground for typography, but it is also a dynamic container that *contextualizes* as well as holds structure.

9. **act as catalysts for reduction of significant form:** Although deletion can be conceptualized as *void substitution* and thus integrated into a standard computational model, **catalytic reduction** which permits void-equivalent forms to be introduced arbitrarily, used to trigger transformation of significant structure, and then freely deleted is an entirely new computational paradigm.

10. **makes rearrangement in space unnecessary:** The side bar illustrates a chain of deduction in text and the same reasoning in the path dialect of iconic logic. The untangling of the path logic is clearly visible.

11. **Truth is confounded with existence:** "The concept of truth can be expressed in an extremely simple and formal manner thus: 'What exists is and what does not does not.'" H. Matsuo (1987) K. Inada (trans.). *The Logic of Unity* p.12.

12. **deviant, or simply wrong, or not even wrong:** A common academic finesse is to leave some problems as *an exercise for the student*. I suspect that the entirety of these three volumes leaves iconic arithmetic as an exercise for any reader.

13. **the ease with which they are retrieved from memory:** Kahneman, p.8.

14. **Varela's embodied cognition, Buddhist and Vedic metaphysics:** See

E. Haeckel (1866) *Generelle Morphologie der Organismen*

C.S. Peirce (1931-1935) *Collected Papers of Charles Sanders Peirce*

L. von Bertalanffy (1968) *General Systems Theory*

G. Bateson (1979) *Mind and Nature*

J. Lovelock (1979) *Gaia: A new look a life on Earth*

I. Prigogine (1980) *From Being to Becoming*

E. Tufte (1983) *The Visual Display of Quantitative Information*

F. Varela, E. Thompson & E. Rosch (1991) *The Embodied Mind*

S. Wolfram (2002) *A New Kind of Science.*

Māyā, the void concept in Vedic texts, has many meanings including *illusion*, the appearance of things that are not present. Buddhist texts as well describe māyā as illusory, a sign without substance.

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Index to the Index

The six **Primary Reference Figures** summarize the structural forms of James algebra and have been isolated from the alphabetical Index as globally applicable. Plurals have been collapsed into their root word. Page ranges separated by double dashes, --, indicate ubiquitous occurrence of a term throughout the range, but not necessarily on every page.

The index is organized by several *semantic reference categories*. The main distinction is between **Symbolic** and **Iconic** concepts. Generally the Iconic Section isolates dominantly iconic concepts. Within the Symbolic Section the separation of algebra from geometry from logic is somewhat arbitrary; these distinctions are made from a perspective of high school mathematics. Another challenge is the categorization of Euler's innovations as algebra or trigonometry or complex numbers, since Euler's equation mixes the three together.

Index

PEOPLE

quotations in bold

PRIMARY REFERENCE FIGURES

TYPOGRAPHIC DELIMITERS

<table>
<tr><th>bracket</th><th>name</th><th>use</th><th>chapters</th></tr>
<tr><td colspan="4">JAMES ALGEBRA</td></tr>
<tr><td>o, ( )</td><td>round</td><td>numeric, exponential</td><td>all</td></tr>
<tr><td>[ ]</td><td>square</td><td>non-numeric, logarithmic</td><td>all</td></tr>
<tr><td>< ></td><td>angle</td><td>reflection, inverse</td><td>all</td></tr>
<tr><td>{ }</td><td>curly</td><td>generic boundary</td><td>32, 34, 37, 44</td></tr>
<tr><td colspan="4">TEXTUAL MATHEMATICS</td></tr>
<tr><td>(.)</td><td>parenthesis</td><td>textual scoping</td><td>all</td></tr>
<tr><td>{,}</td><td>brace</td><td>set delimiter</td><td>33-35, 39-43</td></tr>
<tr><td>{.,.}</td><td>brace</td><td>coordinate pair</td><td>38-41</td></tr>
<tr><td>|.|</td><td>double bar</td><td>absolute value</td><td>32-33, 35, 39</td></tr>
<tr><td>“.”</td><td>quotation mark</td><td>emphasis</td><td>32, all</td></tr>
<tr><td colspan="3">INCIDENTAL</td><td>page</td></tr>
<tr><td>⏋</td><td>cross, mark</td><td>LoF distinction</td><td>xxiv</td></tr>
<tr><td>〔.〕</td><td>shell</td><td>void-equivalent outermost</td><td>147, 380</td></tr>
<tr><td>*[.]</td><td>star square</td><td>bode operator</td><td>124</td></tr>
<tr><td>〘...〙</td><td>double shell</td><td>substitution operator</td><td>120-121</td></tr>
<tr><td>⟦ ⟧</td><td>double square</td><td>two-boundary system</td><td>304</td></tr>
<tr><td>⟨ ⟩</td><td>large angle</td><td>logic, not numeric</td><td>379, 383</td></tr>
<tr><td>{.|.}</td><td>surreal</td><td>surreal number</td><td>85-86</td></tr>
<tr><td>*</td><td>surreal {0|0}</td><td>surreal analog of J</td><td>85-86</td></tr>
</table>

Symbolic Concepts

main topics in bold

GENERAL

ARITHMETIC AND ALGEBRA

GEOMETRY AND TRIGONOMETRY

IMAGINARY AND COMPLEX

INFINITE AND INDETERMINATE

LOGIC AND PROOF

COMPUTER SCIENCE

EDUCATION

HISTORY AND PHILOSOPHY

TYPOGRAPHY

Iconic Concepts

GENERAL

APPLICATIONS

STRUCTURE

BRACKETS

AXIOMS AND THEOREMS

APPLIED PATTERNS

Brackets on the right count demonstrations of each transformation within the volume

Symbols and Icons

definitions in bold
*asterisk * indicates ubiquitous token*

Volumes

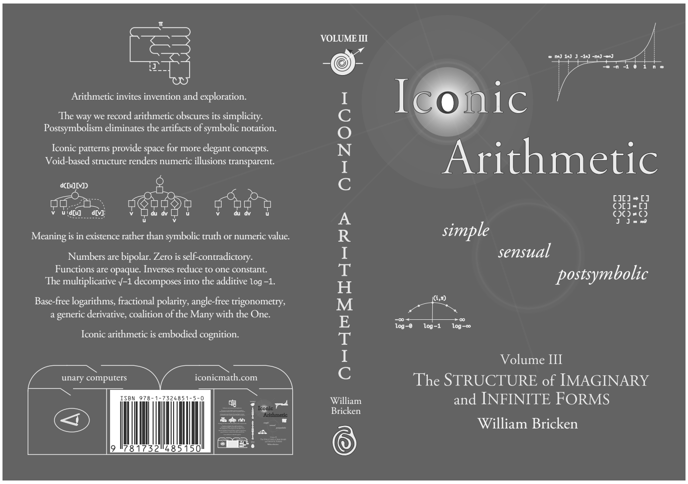
Cover

page 360
the form of π,
room dialect

page 292
complex infinity

page 190
imaginary
exponential

page 367
dnet proof of
product rule
for
derivatives



page 269
unit
axioms

Volume I page 324
Arrangement axiom,
room dialect

page 74
opaque
boundary

page 295
absorption of
polarity at infinity

Cover Words

Arithmetic invites invention and exploration.

The way we record arithmetic obscures its simplicity.
Postsymbolism eliminates the artifacts of symbolic notation.

Iconic patterns provide space for more elegant concepts.
Void-based structure renders numeric illusions transparent.

Meaning is in existence rather than symbolic truth or numeric value.

Numbers are bipolar. Zero is self-contradictory.
Functions are opaque. Inverses reduce to one constant.
The multiplicative √–1 decomposes into an additive log –1.

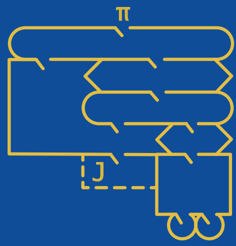
Base-free logarithms, fractional polarity, angle-free trigonometry,
one generic derivative, coalition of the Many with the One.

Iconic arithmetic is embodied cognition.

Bumper Stickers

(some are paraphrased)

WEBSITES



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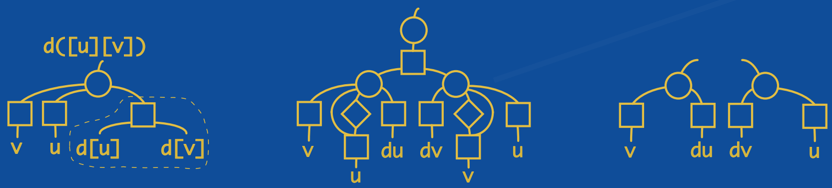

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