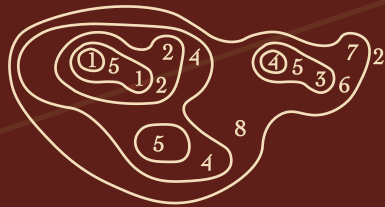
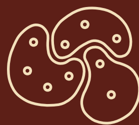
Iconic Arithmetic





simple

sensual

postsymbolic

Volume I

The DESIGN of MATHEMATICS for HUMAN UNDERSTANDING

William Bricken

Iconic Arithmetic

Volume I

Any comments, corrections, refinements or
suggestions you may have will be greatly appreciated.

You can reach me via email at
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Thanks.

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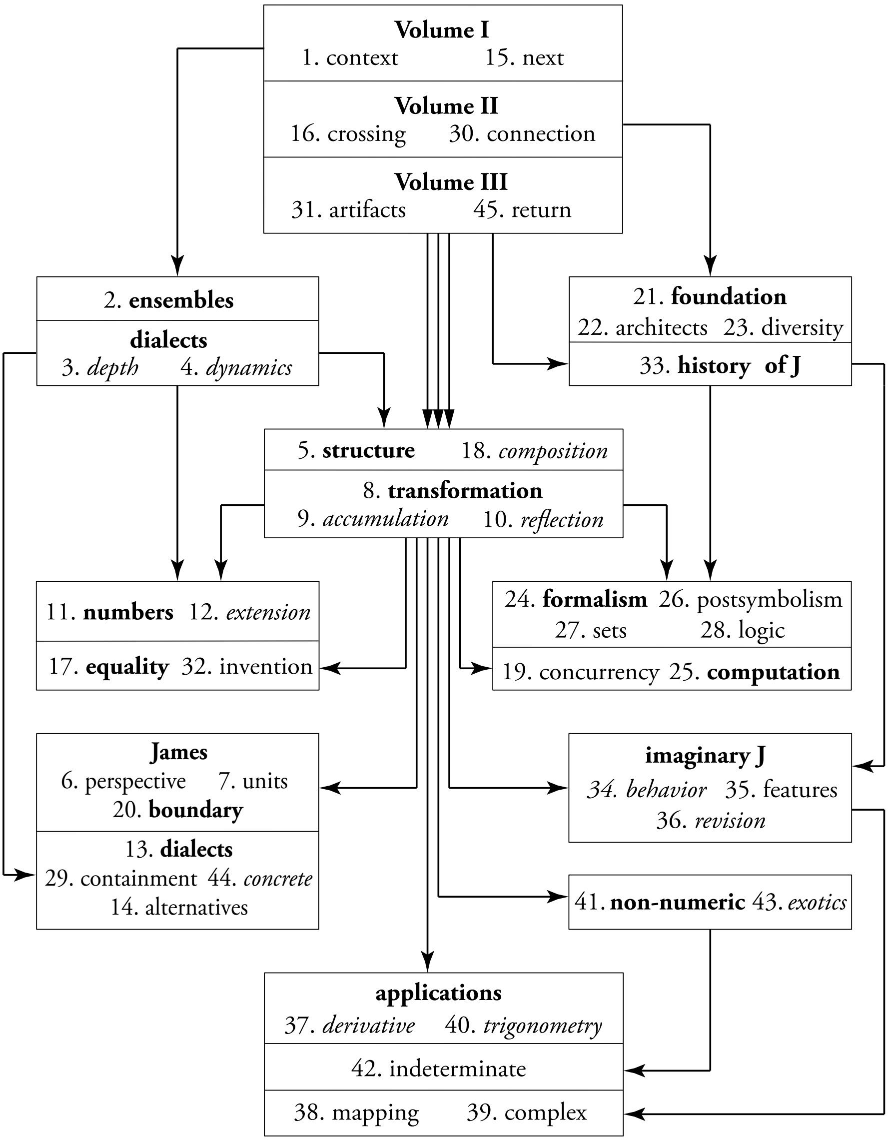
Volume I
The DESIGN of MATHEMATICS
for HUMAN UNDERSTANDING

William Bricken

in memoriam
Richard G. Shoup
George Spencer Brown

Chapters Volume I

Chapter Map



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Preface

The last thing one knows when writing a book is
what to put first.
— Blaise Pascal (1670) Pensées

Perhaps it's not particularly good form to begin with a confession, however: *This volume is a preamble.* My original intention was to explore **iconic logic**. I've elected to write first about arithmetic because not many people are familiar with symbolic logic. Boundary logic might be of interest to only a few and would risk putting the perspectives of iconic mathematics out of reach for many, particularly mathematics educators who rarely see logic within their standard curriculum. Arithmetic however no one escapes, not even preschoolers.

Formal logic has been cherished by Western culture as *the way that the mind works*. Supposedly, critical thinking is built upon logical, rational, formal thinking. Turns out that formal symbol systems have absolutely no correspondence to the way our minds actually work. Human dialogue and human values thoroughly embrace ambiguity, allusion, uniqueness, invention, emotion, multiple entendre, and at times, lawlessness. Studying logic is an arduous excursion into successive abandonment of the biological evolution of human cognition. Few understand how logic arises from nothing, or why we see binary values within an essentially unary form, or how imposed rigor completely avoids temporal and causal and sensual and social and situated information. Given the fundamental role that logic has played in intellectual history, yes our topic should be postsymbolic unary logic. I confess to taking the road more traveled.

George Spencer Brown's seminal work on iconic mathematics, *Laws of Form*, succinctly describes the structural foundations of mathematical thought. This volume is permeated with Spencer Brown's thinking and mathematical wisdom. His text is notorious. The academic analysis, enthusiasm, controversy and rejection of Spencer Brown's work is widely based on a severe misunderstanding that *Laws of Form* describes conventional logic, which it does not. The text becomes much more controversial when it is taken for what it actually is: a postsymbolic foundation for rigorous thinking.

To understand the foundation of mathematics
it is necessary to abandon
the symbolic representation of mathematics.

I've spent over three decades working professionally with boundary logic, yet this volume explores boundary arithmetic. Well, these volumes. Here's a second confession: in order to finish this volume, I had to keep spinning off chapters into secondary storage. The first to go into a separate container was *The Story of J*, which is `[<( )>]` when written as a James boundary form. `J` stands out whenever we do a particular kind of numeric thinking, one that postulates the existence of `–1`. The form of `J` exposes the consequences that follow when we imagine the possibility of bipolar units.

With `J` in its own volume, protecting if you will the rest of numerics from its influence, the content that remains weaves together two themes. There's *what happens* to common symbolic math when it is expressed within a postsymbolic perspective. And there's, um, *how we might think about what happens*. There is a *What!* theme and a *Whaaat?* theme. Both got too big, so the beast split again. Volume I explores the iconic form of arithmetic. Volume II explores the interface between iconic arithmetic and some of our current models of and ideas about mathematics.

Volume I begins with marks drawn in sand, with pebbles held in hand, with notches carved in bone, with unity standing alone. The tally system has been around longer than civilization. Its one-to-one correspondence anchors the birth of mathematical thought. **Unit-ensemble arithmetic** provides the first step toward calculation. Very early in our civilization, humans living in city-states put tallies into groups. These groupings eventually became digits. Throughout most of recorded history counting to determine *how* *many* was a specialist skill. A cupful of beans did not elicit a desire to know how many beans. Numbers were magical icons, only recently do we count. Symbolic arithmetic focuses our attention on the beans, postsymbolic arithmetic focuses on the cup.

Strangely, the accumulation of tallies also gives birth to creatures that are not numbers. *Non-numeric* forms make an early and unavoidable appearance. We are left to contend with the realization that numeric form itself is a subjective imposition upon a broader terrain. Mathematics is about pattern, not number. Volume I builds arithmetic from *a single pattern*, that of **containment**. The non-numeric creatures have been exiled to Volume III.

OK, a third and final confession: it is with trepidation that I comment upon technical masterpieces in the philosophy of mathematics since I'm not a mathematician, have practiced very little abstract math, and have taught even less. I'm a computer scientist. Computer Science is a sister field to mathematics. The structural content it addresses creates a substantively different world-view about symbolic expression and about mathematical abstraction. Volume II particularly stretches into content that I've studied for decades and I still struggle to understand. How, for example, is it possible to separate form from function? How can rigor be independent of reality? How can information be context-free? How can we believe that thinking does not incorporate our senses? How can we pretend that there is a Platonic virtual reality that only our minds can access?

How can any human say that a computer has human characteristics? Computation is quite antithetical to organic existence. Neither does computation trespass into the purely imaginary realms of the infinite. Just like us, computing is embodied, but in a silicon housing that is far from biological. An algorithm has no access to the stuff that dreams are made of. It cannot think, it cannot make distinctions. The Turing test is a direct measure of human gullibility.

I began this project about a decade ago (in the ohohs), in a quite technical vein, wondering whether or not the axiomatic method developed for formal symbol systems could be applied to the simplest tallies used by humanity for thousands of years. Some of this work turned into Chapters 2, 3 and 4. But the motivation arose from a project a decade earlier (the 1990s if you are keeping track). I was working at Interval Research Corporation, Paul Allen's Silicon Valley research lab, with my long-time intellectual companion, Dick Shoup, and a team of inventive scientists including Tom Etter, Fred Furtek and Andrew Singer. Jeffrey James was also on the team for several years, contributing significantly to proof of principle. The *Natural Computing Project* was tasked with this challenge: if you could redesign computing from the ground up, without any consideration for what already exists, what would you build? Dick and I were both theoretical computer scientists interested in foundations, and we both believed that software languages were terribly impoverished, not because they failed to get a CPU to jump through hoops, but because they were built upon baroque mathematical presumptions. Natural Computing developed several alternative mathematics for computation including Shoup's Imaginary Booleans, Etter's Link Theory, Furtek's Torics Dynamic Constraints, and a diversity of supporting FPGA hardware architectures.

Together, Dick and I had both studied George Spencer Brown's seminal work a decade earlier. We understood Spencer Brown's iconic forms not as a path to Eastern philosophy, but as a tool for designing and writing better software languages and for building better computational hardware. During the 80s I met Professor Louis Kauffman. Lou's work conceptually extends *Laws of Form* and I've studied everything he has written. He inspired me to explore rigor creatively; his influence also permeates the mathematical content of this volume. Lou's development of iconic forms for iterated functions, for continued fractions, and for anti-boundaries such as)(, which he calls *extainers*, are excellent extensions of boundary thinking.

I came across *Laws of Form* a decade even earlier (the 1970s), guided by Stewart Brand's review and Heinz vonFoerster's commentary in The Whole Earth Catalog. At that time I was building a home in the forests of Hawaii, and had plenty of time to think about abstraction as I hauled lumber up a hill and pounded 200,000 nails to hold it all together. The unary logic in *Laws of Form* abandons Truth to Existence, a position that seemed quite reasonable to someone who was living in a forest. One day my mother showed up with a copy of *Laws of Form*, saying that she understood nothing inside, but it had literally jumped off the bookstore shelf as she passed by and landed at her feet. She thought it looked like something I might find interesting. Turns out, I'd been reflecting upon Crossing and Calling without the text for over a year. And I knew not to ignore Jungian synchronicity.

Before becoming a software language designer, a decade before those other decades, I was first a teacher, and that's how I earned a living in Hawaii. The very first version of this volume turned into an extended rant about the state of math education in the USA. I was looking for better ways to construct respectful learning environments for growing children and had hoped that taking some of the cruft out of algebra might help. After a few hundred pages of analysis and criticism and ranting, it became obvious that math education is just too easy a target; it is too deeply flawed to justify the effort and the sincerity of a book. Criticism of educational practice implied that I should have been interested in helping to improve educational environments, and that was no longer the case. I had discovered a decade earlier still that *learning* is not within the charter of educational bureaucracy. A school is a place where three or more younger people meet with a state certified older person during specified daylight hours. That's it. I lost interest in schooling just like the Department of Education had lost interest in learning.

I spent the 80s learning CS and AI and ML and VR and UI and CAD and how to talk in acronyms, while becoming proficient in the design of computer languages. I viewed every formal discipline through the lens of boundary mathematics, bouncing ideas and perspectives off of several of the luminaries in Silicon Valley and every one of my professors at Stanford. A dear friend Daniel Shapiro shared many of these avenues of exploration, and both of us edited what the other wrote. My dissertation in part compared the error affordance of conventional and boundary notations for seventh grade algebra students. Net result is that algebra errors made by novices are contextual rather than symbolic, afforded rather than misguided.

We had better unwind. Experience in education led to interest in experiential learning which led to living in a forest which led to embodied math which led to Spencer Brown which led to a Ph.D. about symbolic thinking which led to our group at Interval, protected from the vagaries of underfunded institutions, trying to improve the foundations of computational mathematics. One result might be unexpected: our culture's time-honored and universally accepted place-value number system is not the only way to count, it is not necessarily the best way to do arithmetic, it is not even well designed.

Iconic math is rigorous thinking that looks and feels like what it is intended to mean. Postsymbolic thought is embodied experience. Our topic for the moment then is the deconstruction of common arithmetic based on the formal principles first developed by Spencer Brown, with the American philosopher Charles S. Peirce laying the groundwork at the turn of the twentieth century, and with our nomadic ancestors over 30,000 years ago providing tallies as the original substance from which numbers sprang.

Our ongoing cultural shift from text to imagery, from linear to parallel thinking, from encoded to experiential communication, from reading to watching to participating, suggests a shift from symbolic to iconic mathematics, just the kind of thing Spencer Brown has brilliantly delineated. It was a recent personal discovery that most influenced the evolution of this volume. I learned that marketing modern non-fiction books needs first a documented audience, and that in turn requires hyperactivity (for me at least) in the blogosphere and on Facebook and Twitter. Then I began to run into an embarrassing resistance from the written words. I was quite unable to write for a non-technical audience. In fact, it seemed as though I was unable to write for any kind of audience at all, much less to solicit and cajole that audience into existence. Only after resolutely resigning to my own inadequacies did the writing project become entertaining again.

So, here's a bonus confession: most of what is written herein recounts a conversation I have with myself. Credible fiction is too easy, I much prefer incredible non-fiction. Getting computers to run in unary logic, to abandon the concepts of True and False in favor of present and absent, to protest against models of cognition based upon symbol manipulation was my cup of tea. Arithmetic, something that everyone must deal with, like logic, is in great disrepair. There has been little of substance to improve our understanding of number One and number Two for over a century. If you have math anxiety, if you just don't get it, *it is not your fault*. Mathematics is to blame. Math teachers are asking us to play John Philip Sousa with a broken piccolo.

There are many ways to add 432 to 281. One is to realize that we have in our pockets an exact addition device, one constructed specifically for the task at hand, and all we need to do is to push some buttons. Before digital convergence, this device was called a phone. Another way to add is to incorporate cultural context and not care about exactness. We are putting a bit more than four buckets of a hundred together with a bit less than three buckets of a hundred, about 700 total. Another method is to realize that our culture is constructed so as to protect us from this type of onerous task, and to let a professional behind the counter such as a clerk or a teller do it. Another method is to be a cultural barbarian, a dinosaur. This type of addition is based on an absurd premise: use a paper and pencil and memorized algorithms to seek an exact symbolic number that has no meaning and no context for interpretation, while using a brain that has no evolutionary capacity to do so, and while ignoring our ubiquitous modern computing tools.

Writing words is much easier than implementing software. Words serve as fiction relative to the rigor imposed by automated computation. Dare I add another thread? The actual content of the numerics and logic and programming and education and even writing in this book is focused on *postsymbolic methods of thinking*. What if arithmetic were pictures rather than symbols? Iconic rather than symbolic? Tactile rather than cognitive? Apparent rather than encrypted? Experiential rather than imagined? Concretely finite rather than abstractly infinite? What if numbers were to illustrate rather than encode? Pre-symbolic arithmetic was spatial, visual, tactile and embodied prior to the symbolic reformation of the last century. The totalitarian dictate imposed by the Laws of Algebra has flattened space and touch and perspective and, yes, intelligence into rows of squiggles. Iconic arithmetic is intended to return life to numbers.

So we arrive at the pregnant question: what *should* arithmetic feel like in this century? Exploring and playing with and getting the feel of iconic arithmetic can be astonishingly familiar, it is how arithmetic was before universal schooling sucked the life out of it. If we replace abstraction by embodiment, will mathematics return to Earth? Might we then as a civilization become more aware of and more careful with the only world we have? Is mathematics contributing to the virtualization and then the trivialization of physical reality? Are numbers denizens of cyberspace, sirens that lure us away from stewardship and into fantasy? What will ecological arithmetic look like?

Here's where we stand: three volumes of numerics. The first attempting to save children from symbolic abuse, the second on the formal axiomatics and philosophy of boundary arithmetic, and the third sharing a most exciting feature of the imaginary realm. These three volumes then provide familiarity before wading into unary logic, before the volumes that will explore how to be both rigorous and rational by forgetting what doesn't matter, by ignoring the concept of False as completely irrelevant, by reconnecting rationality to sensory presence, by integrating physical intuition with cognitive visualization, and by treating dichotomy itself as an illusion.

In the text, forward reference to Chapters 16 through 30 refer to Volume II and Chapters 31 through 45 refer to Volume III.

The present goal is to develop an intuitive arithmetic that is both rigorous and embodied, to restore arithmetic to the simplicity of its Babylonian origins as piles of pebbles. The motivation is to provide both school children and adults with an alternative to math-by-symbol-manipulation, an alternative that maintains the essence and rigor of mathematics while also accommodating the essence and sensibility of humanity.

The website `iconicmath.com` is mentioned frequently as a resource for videos and correspondence. It preceded and served as an outline for what you may choose to read next. The Cast of Characters are both colleagues I have learned from and authors I have studied. All are mentioned within the text. This volume does not consolidate what is known nor reflect the views of others. It is an exploration. Welcome to a postsymbolic playground.

william bricken
Snohomish Washington
July 16, 2018

Iconic Arithmetic
Volume I

Chapter 1

Context

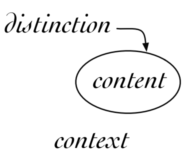
You can recognize truth by its beauty and simplicity.[1]
— Richard Feynman (1985)

We are about to explore **boundary mathematics**, a completely different perspective on the common mathematical tools that we use daily. Boundary math is built from *icons rather than symbols*. **Iconic math** is embodied rather than abstract.

Boundary arithmetic relies upon the single physical relationship of *containment* to express the elementary ideas of arithmetic. In this volume, we will be exploring two types of boundary arithmetic. **Depth-value unit ensembles** unify the way we write numbers with the way we add and multiply numbers. **James algebra** defines the common concepts and operations of arithmetic, (`count`, +, –, x, ÷, ^, `log`, √), as different ways of arranging containers.

The objective is to learn more about conventional arithmetic as it unfolds within the unconventional conceptual system of nested containers.

Modern mathematics is an intellectual infrastructure for solving problems in science, commerce, engineering and technology. It is a vital tool for our civilization. The magnificent edifice of *advanced math* is not in question, since the few who practice advanced math have been extensively trained in navigating the treacherous waters and the precipitous chasms between formal concept and informal communication. Presumably we teach math to everyone because it helps with rational thinking, with formulating a scientific world view, with creating a better world. Since math is a way of thinking, it makes sense to include elements of how humans think into the structure of mathematics, without compromising the formality that distinguishes math from the other disciplines. But **symbolic mathematics**, as currently taught and practiced, is disconnected from human evolution, from human learning, from human psychology and from our natural human capabilities.



Different types of math engender different types of thinking. This volume is also about *a new way of thinking.* Arithmetic serves only as the context for a broader idea. As a small step, if we can come to understand a different kind of math, one that is more natural and more visceral, then we might at least be armed with the knowledge that the math taught in schools is a **design choice**, not a necessity. Although it is possible to read iconic boundaries as common arithmetic, it is also possible to explore iconic modeling without ever considering it to be about numbers or arithmetic or algebra. Boundary math is about **cognitive distinction**.

> *The objective is to learn more about how we think by exploring the formal structure of distinctions.*

If there is a single guiding light it is to recapture the simpler ways of doing arithmetic that evolved in human cultures over millennia, to return to a mathematics that makes sense because it is sensual.

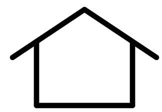
1.1 Communication

A **symbol** is an encoded chunk of information. Symbols are communication tools that have an *arbitrary* representation. Because symbols are encoded, we have to memorize and to recall the patterns that they weave. That's why we have to *teach* children to read and to count. But not to walk, to talk, to see or to think. Symbols specialize in abstract ideas like freedom and ethics. Symbolic description is easily standardized. Symbols also impose a substantial **cognitive load**. It's not a good idea to try to solve a symbolic algebra problem while playing tennis.

symbol

HOUSE

icon



An **icon** *looks like what it means*. It bears a structural resemblance to the ideas it is intended to convey. Icons are communication tools that permit our senses to make the connection between image and meaning. An icon's structure reflects its intention. Iconic arithmetic has a look and feel that connects to our bodies as well as to our minds.

image

The representation of the concept *five* should look like the icon ///// rather than the symbol 5. The concept of *nothing* should not have a representation. The symbol 0 looks like something, it visually contradicts its own meaning. "0" is not nothing.

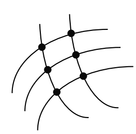
The foundation of iconic arithmetic is the Additive Principle.

Additive Principle

A sum looks like the collection of its parts.

// /// = /////

The principle is physical, based on appearance as well as concept. It is the definition of addition that has been with us since the beginning of civilization. Adding is putting things together. Putting together does not change what things look like. The Principle of Multiplication is that every part of one *touches* every part of another. Multiplication is complete connectivity.



Multiplicative Principle

Every part of one contacts every part of another.

Like any math, iconic math is rigorous. It gives the same results as symbolic math while also maintaining a connection to concrete images and to familiar experiences. As iconic languages, then, James algebra and depth-value unit ensembles have the additional (some would say non-mathematical) requirements of being both **sensual** and **concrete**. The concept called `contains` can be thought, seen and felt.

Three Volumes

arithmetic is putting stuff into boxes

In this volume we'll begin with the arithmetic of tallies, simple marks that generate numbers. We'll see how to add and to multiply tallies, in the process generating **unit-ensemble arithmetic**. We'll use depth-value notation to create parens arithmetic, a typographic yet iconic form. Then we'll expand parens into two and three dimensional spatial dialects and give them dynamics. In Chapter 5 we next explore the structure of James algebra. We'll build common math out of three patterns of containment. Two of these patterns show us what can freely be discarded by calling upon the powerful strategy of **void-equivalence**. We'll look at some examples of experiential dialects of James algebra, and we'll conclude the volume by quickly exploring some other boundary math systems for arithmetic.

Volume II examines the relationship between James algebra and the concepts that currently define formal mathematics. We'll compare boundary mathematics to Frege, Peano, Robinson and other symbolic definitions of number and then make the case for postsymbolic thinking.

Volume III contains a surprise, a reclaimed imaginary number. Conventional analysis focuses on imaginary numbers with their characteristic unit `i`. At the core of James algebra is a new imaginary unit, `J`. As an anchor to familiar concepts, `J` can be interpreted as the neglected `logarithm` of `–1`. The multiplicative imaginary `i` is a *composite* of the simpler **additive imaginary** `J`. In Volume III we'll also explore non-numeric infinite and indeterminate James forms.

Finally a gentle if risky reminder. We will be exploring both the **formal structure** and the conceptual development of common numbers. Our task is to understand what numbers are and how they work by describing them in an unfamiliar foreign language. That language has *making a distinction* (observing a difference) as its primitive operation. Standing back, we'll see that the entire mechanism of arithmetic and algebra can be expressed by three permissions to change structure. Two rules permit structure to be discarded, one permits rearrangement. Standing close, we'll see that the fine-grain structure of arithmetic is incredibly simple.

Once you get comfortable with the idea that structural transformation is independent of interpretation, your eyes and fingers can take over for your brain. Learning to abandon the conceptual and notational mire that we have been taught as common arithmetic no doubt will be a challenge. The most dominant obstacle is our natural propensity to impose previously learned complexity upon the simplicity of void-based reasoning. We'll abandon the use of symbolic expressions in order to see more clearly. We will be looking toward what Jaron Lanier calls *postsymbolic communication*.[2] An irony is that symbolic mathematics is only about a century old: more accurately we are looking backward, to the way that mathematics and particularly arithmetic has been carried out for thousands of years.

The objective is to be able to do pre-college math with our eyes and with our fingers.

1.2 History

Chinese abacus

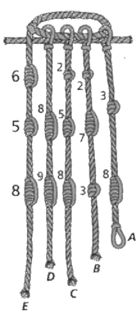
Throughout history, humans have used the abacus, the counting table, the knotted rope, the tally stick and the parts of the body to assist with the tasks of arithmetic.[3] We now have superb digital tools that have replaced physical tools such as pencil and paper. It is time that we end the pretense that people should know the algorithms used

European counting table circa 1550

to calculate with numbers. Addition and multiplication are no longer mental skills, they are buttons on an electronic device.

The brain is the wrong tool to use for calculating.

Incan quipu



For thousands of years, counting in Western cultures (ancient Greek and Roman societies for example) was done by one-to-one correspondence. The introduction of Hindu-Arabic numerals around 800 CE gave numbers their current meaning.[4] Then for another thousand years, until the nineteenth century, counting (and accounting) was done by professional counters, on a counting table, in a counting house. These houses became so popular that they eventually turned into banks.

Classical geometry is a premiere example of an iconic system. The objects of interest, squares and circles and angles, look just like squares and circles and angles. Leibniz, Descartes and Viète converted equations consisting mostly of abbreviated words into symbolic systems that later came into wider use during the seventeenth century. In the mid-nineteenth century the discovery of non-Euclidean systems of geometry contributed to a loss of trust in human mathematical intuition, particularly in our spatial and visual senses. Then we got carried away. Here's mathematician Philip Davis:

Gottfried Leibniz 1646–1716

> For two centuries mathematics has had harsh words to say about visual evidence. The French mathematicians around the time of Lagrange got rid of visual arguments in favor of the purely verbal-logical (analytic) arguments that they thought more secure.[5]

Joseph-Louis Lagrange 1736–1813

The math that we teach in schools lurched into the twentieth century on the back of this crisis in confidence. Those in the mathematical community who wondered about the rigor of mathematics discovered, after thousands of years, that they did not really understand arithmetic or geometry or badly behaving functions or even rigorous thinking. And so the community adopted a radical plan to put mathematics on a firm foundation. The hot idea was **symbolic formalization**, representing concepts using encoded symbols that bear no resemblance to the concepts they identify. Mathematical concepts were to be conveyed using strings of typographic characters rather than using pictures and physical objects and overt behavior. Meaning was to be embedded into the sequential patterns of meaningless squiggles. The disconnection of form and meaning seemed reasonable since, from a Platonic perspective, abstract concepts do not dwell in physical reality. If we cannot point at a concept such as *all numbers* then we surely cannot illustrate it. Any squiggle, perhaps $\forall n$, will do.

David Hilbert popularized among mathematicians the idea that math can be made formal and thus certain by removing all meaning from mathematical symbols, thereby relying solely on the structural relations between symbols. Here's foundational logician Rudolf Carnap:

David Hilbert 1862–1943

> A theory, a rule, a definition, or the like is to be called formal when no reference is made in it either to the meaning of the symbols (e.g. the words) or to the sense of the expressions (e.g. the sentences), but simply and solely to the kinds and orders of the symbols from which the expressions are constructed.[6]

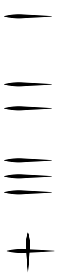
In an ironic twist, Hilbert then argued that arithmetic carries its own meaning.

> In elementary domains of arithmetic...there is that complete certainty in our considerations. Here we get by without axioms, and the inferences have a character of the tangibly certain.

Hilbert continued that the essential relationship between the representation of a number and the concept of a number is iconic, that

> *the object doing the representing contains the essential properties of the object to be represented.*[7]

Brahmi digits circa 800 CE



Hilbert was talking specifically about putting iconic strokes together to yield sums (e.g // + /// = /////), the unit-ensemble model described in Chapter 2. Thus we are in complete agreement with Hilbert: numbers are essentially iconic.

Symbolic math was *invented*, along with the horrible design idea that math should be done with our minds and our memories, rather than with our eyes, our bodies and our physical tools. The unifying theme was that every mathematical object is a *set*. The goal was to endow math with **purity**, to collapse all of mathematics into one grand scheme. Assumed invariant patterns called **axioms** defined how to think rigorously. Nicholas Bourbaki:

Bourbaki School 1935

> The internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts...and which has led to what is generally known as the "axiomatic method."[8]

The great success of the entirely symbolic approach, followed closely by the rise of the use of streams of ones and zeros in digital computers, has led to the expression of mathematics almost exclusively in symbolic string languages. Unfortunately, in the rush to make math symbolic, the early twentieth century founders seem to have forgotten that people care about *understanding* and *using* math, not about an esoteric collection of structural rules that cast common arithmetic into conformity with a maze of symbolic concepts delineated by group theory built on top of set theory built on top of predicate logic. Logicians Barwise and Etchemendy observe:

> Despite the obvious importance of visual images in human cognitive activities, visual representation remains a second-class citizen in both the theory and practice of mathematics.[9]

Belief

Mathematics is a human endeavor, obviously. Mathematics is replete with metaphysical concepts (also called *beliefs*) such as completed infinity, physical continuity, Platonic virtual reality, universal and eternal truths, constructions so grand that they are beyond the scope of time and space, extra-human origins of human ideas, calculations that may not halt, and formal objects that exist but cannot be identified. These belief systems are not necessary for a formal description of nor for an informal understanding of arithmetic.[10]

Perhaps there is not one grand system that unifies mathematics. As scholars studying ethnomathematics have affirmed, mathematical thought exhibits organic diversity.[11] Here is computer scientist Joseph Goguen's attempt to disperse the totalitarian attitude that mathematical thinking is uniform and universal:

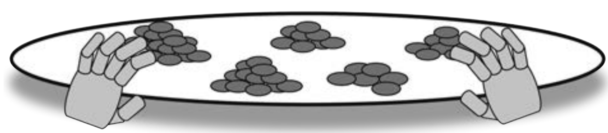
Joseph Goguen
1941–2006

> Notation is only the surface reflection of these deeper, essentially social struggles. But perhaps it is time we realize that no metanarrative can be demonstrably superior to all others for all purposes, and that, in this sense, we live in many different worlds, rather than in just one world.[12]

A premiere example of a physically dysfunctional symbolic concept is the associative law of addition.

$$(a + b) + c = a + (b + c)$$

The parentheses used to assert this law create visual interference that undermines the meaning of addition, that of **putting together**. Putting things together two-at-a-time is not a property of addition. The associative law specifies a *method* to achieve addition, one that is not particularly efficient. The sequential two-at-a-time strategy might be because there are two sides to a textual plus sign; it might be because addition tables are constructed to add only two numbers at a time; it might be because relations are usually binary; it might be because group theory incorporates right- and left-side rules; or it might be a hangover from counting by adding one at a time. But here's what we can learn from children. *To add many things, put them together. It doesn't matter how.*



Vision

The earliest founders of rigorous mathematical systems used diagrams and visual thinking extensively. Venn and Frege and Peirce all developed functional visual mathematical notations. They understood that thinking unites imagery with structure. According to Charles S. Peirce,

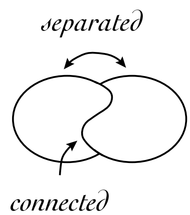
Charles Sanders Peirce
1839–1914

> All necessary reasoning is diagrammatic. ... all reasoning depends directly or indirectly upon diagrams.[13]

Many rigorous iconic calculi have recently been studied; these include fractals, cellular automata, particle diffusion, knot theory, and the paradigm of modern algebra, category theory. Philosopher Danielle Macbeth:

> At least for the case of mathematical concepts, then, we can say exactly what meaning is: it is nothing more and nothing less than what is exhibited in... diagrams and expressions that directly display the senses of concept words, senses within which are contained everything necessary for a correct inference.[14]

Mathematics is necessarily sensual, so let's pardon it from banishment into symbolic disembodiment. Let's ground arithmetic in the tangible Earth rather than in imaginary realms of conceptual abstraction and let's require that it recognize reality as we humans experience it.

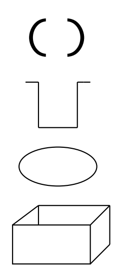


1.3 Boundary Forms

Boundary forms are *configurations of nested boundaries*. Boundaries both separate and connect. It is convenient to consider these boundaries as **containers** that have an inside and an outside. In this volume, we'll turn the concept of containment into a comprehensive mathematical tool.

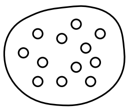
In James algebra for example *everything* is a container. Empty containers are units. There are two basic forms: ( ) and [ ]. They cancel each other out. When we wish to connect to arithmetic, ( ) is One and [ ] is a type of Infinity. When we wish to think about cognitive distinctions, we might say that ( ) is *what is* and [ ] is *what could be*. When we wish to eschew interpretation, the name of ( ) is **Round** and the name of [ ] is **Square**. These delimiters and their names are quite arbitrary, dictated by the available keys on a typewriter, and having no relationship to the geometric shapes they may invoke. These delimiters represent generic containers. The only thing that a container does, regardless of shape or representation, is to hold things.

some containers



A central innovative idea is to treat non-existence, *void*, with the respect it deserves. *Void* has no properties and does not exist, taking the concept of 0 with it into oblivion. An **empty container** is *void* inside. Empty containers are **units** precisely because they hold nothing inside.

an ensemble



A collection of units within a common container is an **ensemble**. We can interpret the *cardinality* of an ensemble as a **number**. Addition is eliminating the boundaries that distinguish separate ensembles. Multiplication is putting ensembles inside other ensembles. Addition eliminates distinctions while multiplication nests them.

James algebra includes a **generalized inverse**, < >, which is a *single* boundary concept that condenses the basic inverse operations of arithmetic. Negative numbers, subtraction, unit fractions, division, roots and logarithms are essentially the *same operation* applied in different contexts. Cognitively, < > can be associated with *reflection*, with turning structure back upon itself. It's name is **Angle** and alone it is nothing.

Containers

The concept of a container is tangible, not abstract. *All boundary forms can be constructed in physical space.* Containers are fundamentally different than symbols, containers have a physical presence. Containers also have an *inside*. Here's Lakoff and Núñez as they develop a theory about the cognitive origins of mathematics:

> The Container schema has three parts: an Interior, a Boundary, and an Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole.[15]

Containers are so useful that they form the structural basis of

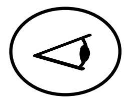
— computer circuity and logic
— mathematical sets and functions

— web description languages like HTML and XML
— software design environments like InDesign
— storage closets
— loading docks
— suitcases and
— kitchens.[16]

A container distinguishes two spaces, its inside and its outside.[17] It is an object from the perspective of its outside **context**, and an operator upon its inside **content**. When we look at a container from the outside, it appears to be an independent object. When we are inside a container, it feels like an environment.[18] When we make a distinction, we are its container. It is *our viewing perspective* that determines whether a container is an object that can be manipulated or an environment that can be experienced or a concept that can be imagined.

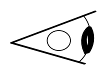
object

environment



Boundaries invite participation. This feature is sufficiently important to elevate it to our first principle, the Principle of Participation.

distinction



Participation

How we look at a form determines what it is.

There is *only one relationship* between two containers, `A contains B`. Containment, together with the structure of boundaries, is sufficient to express completely the objects and operations of elementary arithmetic and algebra. This is what it means to *simplify* arithmetic: to describe it fully using one idea, the **distinction**, as represented by a container that distinguishes inside from outside.

`A contains B`
$(B)_A$

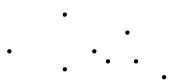
Containers find their cognitive expression as *categories,* which are firmly grounded in the sensory-motor systems of the brain. Lakoff and Núñez again:

> Propositional calculus (the simplest form of symbolic logic) is ultimately grounded in Container schemas in the visual system.[19]

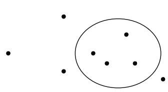
From an abstract perspective, the name "contains" is arbitrary, a metaphor for visualization but not essential. We are exploring a system with one **binary relation**, `R`, that has the properties of being irreflexive, asymmetric, and intransitive. These properties are more or less the opposite of the properties of `equals`. In other contexts, `R` might be called `parent`, `underneath`, `eats`, `causes`, `implies`, and an unlimited number of other evocative names.

Distinction

Well, there is a refinement to be made. The idea of a "binary relation" comes from our conventional foundation of logic and sets. Relational concepts such as reflexive and symmetric and transitive are themselves derivative rather than definitional. We are beginning *prior to* logic, prior to the formal concept of a relation. The abstraction that generalizes the `contains` relation is **distinction**, or *difference*. Spencer Brown provides a definition:



distinction



> A distinction is drawn by arranging a boundary with separate sides so that a point on one side cannot reach the other side without crossing the boundary.[20]

Distinction is not numeric, it has no units of measurement and no scale of comparison. "Difference is not a quantity."[21] As such, distinction is not grounded in science, it is grounded in experience. Gregory Bateson continues:

Gregory Bateson
1904–1980

> A difference is an elementary idea. It is of the stuff of which minds are made.[22]

Our minds and our senses transact differences. Matisse: "I don't paint things. I only paint the difference between things."[23]

An algebra of distinctions is fundamentally different than both an algebra of measurements and an algebra of numbers. Actions in the world involve energy. We might say that they occur *on the outside* of the boundary of our bodies. In contrast, difference does not involve energy. Difference is not localizable, it cannot be assigned a place or a time. Difference is *on the inside*. When we compare a duck to a chicken, the difference does not reside within the duck, nor is it within the chicken, nor is it within the space that separates the two. "In a word, *a difference is an idea*."[24] Let's elevate Bateson's definition to our second principle, the Principle of Distinction.

difference is dimensionless

Distinction

Difference is an idea.

Our bodies then are the concrete realization of physical difference, while our thoughts are the abstract realization of virtual difference. Bateson provides clarity,

> The explanatory world of *substance* can invoke no differences and no ideas but only forces and impacts. And, per contra, the world of *form* and communication invokes no things, forces, or impacts but only differences and ideas.[25]

Without distinction, there is the **non-concept** *void*. This underlying page supports typographic characters. Similarly *void* is the substrate that supports distinction. Like the white space of the page, *void* is everywhere. Unlike this page, *void* has absolutely no properties of any kind. This means that there are no relationships between containers within the same space since there is no intervening medium to support a relationship. No points, no distances, no coordinates, no dimensions, no metrics. It is as if the characters printed on a page were each independent and unrelated to one another. All that they would share is the page or the line they are recorded upon. There is only containment. Our third principle:

Void

Void has no properties.

1.4 Embodiment

Our senses and our bodies are the interface between manifest and conceptual. Distinction is the ground of perception. When there is no distinction between inside and outside, we do not perceive a difference. The interface between our physical and our cognitive selves is a boundary that transacts only differences. The physicality of our body defines an obvious container of our self. Boundary math carries this physicality, via iconic representation, directly into the core of mathematical thinking. Philosopher Mark Johnson:

> The container schema's structural elements are "interior, boundary, exterior," its basic logic is "inside or outside," and its metaphorical projection gives structure to our conceptualizations of the visual field (things go in and out of sight), personal relationships (one gets in or out of a relationship), the logic of sets (sets contain their members), and so on.[26]

James algebra and unit ensembles are both **embodied**. The *concept* of a container is necessarily abstract. The representation of a container is physical, a tangible manifestation of the concept. The meaning of the representation is unequivocally an idea, an idea that can at any time be enacted. Lakoff and Núñez: "Ultimately, mathematical meaning is like everyday meaning. It is part of embodied cognition."[27]

nominalism
abstract objects do not exist

Mathematics and mathematical "truth" necessarily co-evolve, not only in a conceptual sense, but also in coordination with the physical evolution of our brains and our bodies. We come to understand mathematical ideas *through* our bodies. Bluntly, mathematics is "conceptualized by human beings using the brain's cognitive mechanisms."[28] Containers are intended to re-anchor mathematics to physical existence, to return the beast to its creators, to tame those aspects of the beast that we might encounter in daily life. We have evolved from mathematical Platonism to mathematical nominalism.

Rigor

Rigorous math requires that mathematical techniques be independent of our personal and cultural biases and beliefs. We want math to be extremely useful and completely reliable and also uncompromising in its neutrality. But does neutrality necessarily imply non-human? Must math be an abstract fantasy removed from human existence? Can math be both rigorous and organic?

In *Proofs and Refutations* Lakatos chronicles the evolution of a single formal idea, the structure of polyhedra. He concludes:

Imre Lakatos
1922–1974

> Mathematics, this product of human activity, 'alienates itself' from the human activity which has been producing it. It becomes a living, growing organism, that *acquires a certain autonomy* ... its own autonomous laws of growth, its own dialectic.[29]

It is time to reclaim mathematics as a human activity, to tame the creature we have unleashed. **Humane math** aligns with how our minds are known to work, with how the patterns of communication between people work, and with how physical existence embodies our knowledge. *Boundary math is both humane and rigorous,* the two objectives are not incompatible. The result is a much simpler iconic arithmetic that reflects how numbers evolved in human cognition prior to our recent truly exotic experiment with purely symbolic mathematical form.

Truth and Beauty

This exploration has deep parallels with the way that physicists describe reality. The basic technique of Physics is to derive simple mathematical models that condense disparate observations. Here we are exploring the basic nature of arithmetic by developing a simple model that condenses disparate algorithms and operations. Einstein:

> A theory is more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability.[30]

Two different types of problem might arise. Further observation may introduce new data that is inconsistent with the model. In this case the model needs to be expanded, or perhaps abandoned altogether. Problems can also occur *within* the model itself. Perhaps a nasty division by zero might be encountered. An intractable infinity might be swept away through an inventive mathematical technique such as renormalization. Or the model might be abandoned altogether. Such was the case in the early nineteenth century when Lobachevsky and Bolyai (and Gauss before them) developed geometries that abandoned the parallel lines axiom of ancient Greek geometry.

It was Maxwell's electromagnetic equations in the late nineteenth century that first challenged the idea that physics is about perceived reality. His wave equations ignored both the how (i.e. the physics) and the what (i.e. the observable). Mathematician Zvi Artstein:

> The only justification for the existence of the waves was the facts that they provided a solution to the equations, solutions that have wavelike characteristics....He forwent the physical explanation based on known physical quantities. He published his equations, as if declaring this is physics, the physics is inherent in mathematics.[31]

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

When Michaelson and Morley went looking for the aether that supported the wave propagation of light they found nothing. There was no physical substrate.

Maxwell's equations are beautifully symmetric. Beautiful models, those with lots of symmetry and regular structure, have been preferred historically over models that are messy. Scientists Augros and Stanciu: "Beauty is the primary standard for scientific truth."[32] The same is true of mathematics. Here's mathematician Godfrey Hardy:

> Beauty is the first test: there is no permanent place in the world for ugly mathematics.[33]

This despite the possibility that both Nature and mathematics may at their core be messy. To maintain mathematical beauty, new, never before observed phenomena might be postulated. Experiments are then designed to expose the potential new phenomena. Theory drives experiment. Often different theories diverge, to arrive at completely different *descriptions* of the same observations. In this case, attempts are made to merge the theories, or perhaps the theory with the greatest explanatory power is kept and the others tossed away. Physicist Murray Gell-Mann:

> Frequently a theorist will throw out a lot of data on the grounds that if they don't fit an elegant scheme, they are wrong.[34]

What is *true* is moot, since we can also be guided by what is *simplest*. The field of mathematics has recently abandoned the connection of Truth to Reality by defining its own truth as its own internal consistency. The concept of Truth itself can work against accurate description, as it does in the case of the wave/particle duality in quantum mechanics. Physicist Paul Dirac:

> It is more important to have beauty in one's equations than to have them fit an experiment.[35]

This attitude stems from a growing realization that modern physics is no longer connected to our perception of Nature. Mathematical physicist Roger Penrose:

> It is a common view among many of today's physicists that quantum mechanics provides us with *no* picture of 'reality' at all! The formalism of quantum mechanics, on this view, is to be taken as just that: a mathematical formalism.[36]

When physics becomes mathematics, our concepts of physical reality begin to conform to the values of mathematics. Elegance, symmetry, scope, these are how leading scientists view Nature. Here's Stephen Hawking:

> I don't demand that a theory correspond to reality because I don't know what it is. Reality is not a quality you can test with litmus paper. All I'm concerned with is that the theory should predict the results of measurements.[37]

And we know from quantum mechanics that the results of measurements, what we see, depend upon how we elect to look. Reality cannot escape its sensory basis.

1.5 Humane Mathematics

If metaphysical Truth and physical reality are no longer the criteria for knowledge, are there multiple ways to describe the concept of number? Which are preferable? Specifically, which description of numbers, a beautiful concise one or a messy complex one, is preferable? Are any particular models of mathematical patterns more true than others? Is the way we write arithmetic fully separate from what we are attempting to express? Is the barrier between representation and meaning an impermeable wall, or is it perhaps a bridge, or is it a fantasy of convenience that supports meta-analysis at the cost of comprehension?

In the realm of physics, experiment and observation often help to guide the choice of language for a theory. In elementary mathematics, the ability of children to learn simple concepts can be taken as experimental evidence. A mathematical concept is not simple if kids do not understand it. Our current observational data is that after studying arithmetic for a dozen or so years, about two out of every three children and young adults in the U.S. do not understand such elementary numeric ideas as negative numbers, fractions and the rule of distribution that connects addition and multiplication. These three concepts account for the vast majority of student errors in math classes.[38] Might the wisdom of youth be telling us that perhaps we have the wrong model of arithmetic itself? Cognitive scientist Stanislas Dehaene:

> When we think about numbers, or do arithmetic, we do not rely solely on a purified, ethereal, abstract concept of number. Our brain immediately links the abstract number to concrete notions of size, location and time.[39]

We have a choice to continue to support a mathematical and symbolic perspective that denies human experience and the reality of our physical presence in a physical environment, or to encourage students to be aware and responsible within the world of physicality by teaching mathematics as a pragmatic tool rather than as an abstract, cognitive skill.

Arabic digits circa ~800 CE

١
٢
٣
٤

The vision is to return to **native mathematics**, prior to the conversion of the visceral sensate numbers of our cultural and physical evolution into the meaningless symbolic squiggles of the twentieth century. Symbolic math may have been completely appropriate during the transitional period from about 1910 to 1980, prior to the computer revolution. Now it is time to give symbolic processing to its rightful owners, those microscopic arrays of transistors and wires that constitute digital hardware. It is time to return mathematical concept, understanding, intuition, and just plain common sense to their rightful owners, flesh-and-blood humans. We can reserve symbolic math for professionals, and for students who declare their interest in the field of mathematics as a college major. We can maintain **formalism**, the essence of modern mathematics, within precollege math without loosing our humanity by embodying it in utility. We do not need to pretend that math exists in some abstract virtual world. We can own math as a human invention, a tool to aid thought.

We do not need math classes for non-mathematicians.

Any subject can call upon the tools provided by a mathematical perspective, when needed, when appropriate and when useful. All it takes is recognition of the obvious: symbolic math is not humane math. Ask any school child.

Simplicity

What is necessary first is to show that non-symbolic math does exist, that we can anchor the formal concepts of arithmetic within common sense and within common sensation. Here's Stephen Wolfram:

> Like most other fields of human enquiry mathematics has tended to define itself to be concerned with just those questions that its methods can successfully address…the vast majority of mathematics practiced today still seems to follow remarkably closely the traditions of arithmetic and geometry that already existed even in Babylonian times.[40]

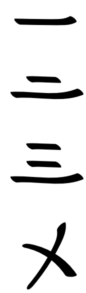
There is an organic legitimacy, a human context, from which mathematics arose. But we became enamored by the symbolic process, without adequately recognizing the cost. Historian of mathematics Alberto Martínez comments:

> Even when elementary mathematical rules are designed to represent plain manipulations of things, they might still be used to construct symbolic statements that do not correspond to anything that can be exhibited palpably.... Throughout history, mathematicians realized that by adopting diverse and particular empirical explanations to justify specific symbolic operations, mathematics acquired a semblance of arbitrariness and inconsistency. Thus they came to cast aside empirical explanations as mere illustrations and applications, and not as justifications for mathematical rules.[41]

The concept of truth has migrated from experience to symbolic structure. Mathematics abandoned the physical in favor of its own internal structure. In the first half of the twentieth century, however, it became clear that mathematical structure had neither global truth nor consistency to offer. Martínez again:

> Virtually nobody imputes to traditional algebraic methods any blame for instances where the symbolism generates unsatisfactory or bizarre results... virtually nobody says that maybe it is the algebra that is defective.[42]

Suzhou digits circa ~1200 CE



Professional mathematicians are trained to use powerful and exotic tools, and advanced math is such a tool in the hands of a professional mathematician. But when, for example, should an average driver know how to fix an electronic carburetor? When should a parent know how to prepare *quiche lorraine* for breakfast? When should a web surfer know how to write XML database search algorithms? When should a homeowner use the quadratic formula?

The *discipline* of math is built upon absolute abstraction. It is intentionally and adamantly disembodied (ignoring the human), ungrounded (ignoring the earth), and imaginary (ignoring reality). Philospher of mathematics Brian Rotman observes that mathematics itself is built upon

> A world of mathematical objects — numbers, points, lines, sets, functions, morphisms, spaces, and the like — that are held to exist prior to and independent of any talk, description, or discussion of them....the belief in objects "out there" — uncorrupted by the vagaries and uncertainty of history, culture, human choice, and the associated subjectivities that permeate discourse — is crucial and nonnegotiable.[43]

The concepts that constitute abstract math (symbolic manipulation, Platonic reality, infinity, zero) are *theological*. It is churches that manipulate sacred symbols as doctrine; it is religions that call upon an imaginary reality for miracles; it is only gods that are infinite. Math has established itself to be sacred, to be beyond humanity, and thus to be beyond common comprehension. Unlike religion, however, one's belief in math is not rewarded by absolution or by contentment or by virtue. Failure to believe in the whole numbers leads to extreme derision. A numeric atheist (a matheist) gets condemned not to hell but to ignorance. While common religion allows the needy the salvation of belief, common math simply equates need with incapability.

European digits circa 1900 CE

1
2
3
4

The **Doctrine of Abstraction** is unique to Western academic mathematical thinking. Street merchants in Brazil, money lenders in India, school children in Africa, grocery shoppers in an American supermarket, even statistical scientists measuring the popularity of a political candidate, none comprehend numbers as abstractions. Their real-world numbers are constantly grounded in application, in utility, in human dynamics. **Ethnomathematics**, the study of how people actually use math, sees the extreme abstraction embodied in Western thought as yet another mechanism of cultural imperialism.[44]

The mathematics of chaos theory, iterated functions and cellular automata each embody an anti-abstraction principle. Although computation in these fields can be deterministic, the only way we know what the next step will bring is to compute that step. It is not possible to develop a symbolic abstraction or condensation that models or that predicts what is next. There is nothing simpler than real-time unfolding.

Yes, it will probably be very embarrassing for our culture to admit how delusional it has been, but by now we should be accustomed to delusional error. No, the Earth is not the center of the universe. Those twinkles in the sky are not all stars. Indigenous peoples are not savages. Obviously all mammals have emotions. The universe is not made of matter, nor is it made of mathematics. Mathematics itself is the tool that allows us to explicitly ignore the deeper detail of our world in favor of a crisp abstract summary that we can see, handle and use.

1.6 Remarks

Symbolic math is a big part of the problem of math education; another big part is our cultural belief that all people should learn the details of an esoteric discipline at the cost of their own self-confidence. We may find that symbolic abstraction is a weakness as well as a strength. When it comes to problem solving, we might find that symbolic arithmetic is an antiquated tool. We might find that iconic arithmetic provides greater elegance, simplicity, groundedness and hopefully ease of learning.

In the rest of the volume, we will be seeing just how much of algebraic theory can be abandoned without loss, in pursuit of reconnecting mathematics with humanity. **Notation**, how we record mathematical ideas, is emphasized as both a problem and a solution.

In the text we will often transcribe between two different representational (and conceptual) systems, between conventional strings and boundary math containers. I'll introduce a transcription symbol, ☞, to indicate when we are changing systems.

one formal system ☞ *another formal system*

We will restrict the use of ☞ to cases in which there is a specifically known **transcription map** between two systems, a map in which transcription does not introduce confusion. This makes ☞ a type of equal sign. We'll call it "*the finger*". The pointing finger is also a reminder that a *cognitive shift* is needed. Transcription is broader than mapping the same concept across two different notations. Underlying the profound difference between symbolic and iconic notations is a substantively different system of concepts about what arithmetic is and how it works. The finger, ☞, makes us aware that transcription from conventional expressions to boundary forms includes eliminating the conceptual infrastructure of sets, functions and logic.[45]

simple ≠ familiar

In *A New Kind of Science*, Stephan Wolfram valiantly reconstructs Science itself without calling upon calculus or algebra or arithmetic or logic.

> The presence of logic is in fact not essential to many overall properties of axiomatic systems.[46]

James algebra follows distantly in Wolfram's footsteps to construct a new kind of arithmetic. As you might expect, at first glance James patterns appear to be exotic (just like any math). The initial learning curve is not steep, but the new terrain may be quite unfamiliar.

To establish a common ground, in Chapter 2 we'll first visit a system that is familiar to us all, **unit-ensembles**. Unit-ensemble arithmetic describes how creatures like ///// behave. Then we'll tame the multiplicity of tally marks with the tools of depth-value in Chapter 3, and provide many sensory and dynamic depth-value representations of arithmetic in Chapter 4.

In Chapters 5 and 6, we'll become oriented to James algebra and the primary conceptual content of this volume. Chapters 7 through 12 address James algebra from a structural perspective. How can only containment express all of arithmetic? What makes a unit? How does structural change work? How does counting work? How do numbers work? In Chapters 13 and 14, we'll return to higher dimensions to provide several examples of sensory and experiential systems of James arithmetic.

Endnotes

1. **opening quote:** R. Feynman (1985) in K. Cole *Sympathetic Vibrations: Reflections on physics as a way of life.*

2. **looking toward what Jaron Lanier calls *postsymbolic communication*:** J. Lanier (2010) *You are Not a Gadget* p.190.

Lanier is motivated by the potential of virtual reality, a more visceral but still virtual interface with concept. "We'd then have the option of cutting out the "middleman" of symbols and directly creating shared experience. A fluid kind of concreteness might turn out to be more expressive than abstraction."

3. **to assist with the tasks of arithmetic:** Images on these pages are from the Computer History Museum, `http://www.computerhistory.org/revolution/calculators/1`

4. **that gave numbers their current meaning:** D. Schmandt-Besserat (1992) *How Writing Came About*.

5. **arguments that they thought more secure:** P. Davis (1997) Mathematics in an age of illiteracy. *SIAM News* **30**(9) 11/97 p.30.

6. **solely to the kinds and orders of the symbols from which the expressions are constructed:** R. Carnap (1937) *The Logical Syntax of Language* p.1.

7. **the essential properties of the object to be represented:** Hilbert's quotes are from M. Hallett (1994) Hilbert's axiomatic method and the laws of thought. In A. George (ed.) *Mathematics and Mind* p.184-185. Italics in the original.

8. **what is generally known as the "axiomatic method":** N. Bourbaki (1950) The architecture of mathematics. *American Mathematical Monthly* 57 p.222.

9. **both the theory and practice of mathematics:** J. Barwise & J. Etchemendy (1996) Visual information and valid reasoning. In G. Allwein & J. Barwise (eds.) *Logical Reasoning with Diagrams* p.3.

10. **not necessary for a formal description of nor for an informal understanding of arithmetic:** Throughout history mathematics has been associated with the work of God, although it is difficult to separate any of

the activities of Western culture from attribution to a Christian diety. Philip Davis has a nice overview.

P. Davis (2004) A brief look at mathematics and theology. *The Humanistic Mathematics Network Journal Online* v27. Online 2/2017 at `https://cs.nyu.edu/davise/personal/PJDBib.html`

For incisive analysis, see B. Rotman (1993) *Ad Infinitum The Ghost in Turing's Machine: Taking God out of mathematics and putting the body back in*.

11. **mathematical thought exhibits organic diversity:** Ubiratan D'Ambrosio pioneered the idea that different cultures have different forms of mathematical thinking, contrasting the Western scientific method to more native styles of thought in South America. In A Histographical Proposal for Non-western Mathematics D'Ambrosio focuses on

> the social, political and cultural factors in the dynamics of the transfer and the production of scientific and mathematical knowledge in the colonies, as well as on the recognition of non-European forms of science and mathematics.

Quote in H. Selin (ed.) (2000) *The History of Non-western Mathematics* p.79-92.

A good analysis is in R. Vithal & O. Skovsmose (1997) The end of innocence: a critique of 'ethnomathematics'. *Educational Studies in Mathematics* **34** p.131-158.

12. **we live in many different worlds, rather than in just one world:** J. Goguen (1993) *On Notation*. Department of Computer Science and Engineering, University of California at San Diego.

13. **all reasoning depends directly or indirectly upon diagrams:** C.S. Peirce (1904) The new elements of mathematics. In C. Eisele (ed.) (1976) *Mathematical Philosophy* v4 p.314.

14. **within which are contained everything necessary for a correct inference:** D. Macbeth (2009) Meaning, use, and diagrams. *Ethics and Politics* **xi**(1) p.369-384.

15. **the parts make no sense without the whole:** G. Lakoff & R. Núñez (2000) *Where Mathematics Comes From: How the embodied mind brings mathematics into being* p.30-31.

16. **loading docks, suitcases, and kitchens:** In *Metaphors We Live By* (2003) G. Lakoff and M. Johnsen extend the CONTAINER metaphor to include houses, words, linguistic expressions, argumentation, forests, clouds, sports events, social groups, territoriality, time, activity, our visual field, our bodies, and our lives.

17. **its inside and its outside:** The Jordan curve theorem asserts that a closed loop on a flat surface does construct an inside and an outside. Its symbolic proof is notoriously difficult, presumably because the obvious properties of spatial forms are obscured by strings of symbols.

An engineer, a physicist, and a mathematician were faced with the problem of putting a herd of sheep inside a fence while using the least amount of fencing materials. The engineer rounded the sheep up into a tight group and then put a circular fence around them, declaring that the circle encloses the most area for the least fencing. The physicist imagined a fence with infinite radius, and then tightened it around the sheep, declaring that this fence is most efficient. The mathematician built a small fence around himself and declared: "I am on the outside".

18. **When we are inside a container, it feels like an environment:** Consider an automobile. When we stand outside it is a beautiful, or perhaps a utilitarian, object. When we drive, the automobile becomes an encompassing environment that we feel to be separate from the outside.

19. **Container schemas in the visual system:** Lakoff & Núñez, p.134.

20. **cannot reach the other side without crossing the boundary:** G. Spencer Brown (1969) *Laws of Form* p.1.

21. **Difference is not a quantity:** G. Bateson (1991) *A Sacred Unity* p.219.

22. **It is of the stuff of which minds are made:** Bateson, p.162.

23. **I only paint the difference between things:** attributed to Henri Matisse without a specific reference that I could locate.

24. **In a word, a difference is an idea:** G. Bateson (1972) *Steps to an Ecology of Mind* p.481.

25. **but only differences and ideas:** Bateson, p.271.

26. **the logic of sets (sets contain their members), and so on:** Paraphrased from M. Johnson (1987) *The Body in the Mind*. In F. Varela, E. Thompson & E. Rosch (1991) *The Embodied Mind* p.177.

27. **It is part of embodied cognition:** Lakoff & Núñez, p.49.

28. **using the brain's cognitive mechanisms:** Lakoff & Núñez, p.3.

29. **its own autonomous laws of growth, its own dialectic:** I. Lakatos (1976) *Proofs and Refutations: The logic of mathematical discovery* p.146.

30. **the more extended is its area of applicability:** A. Einstein, in P. Schilpp (ed.) (1979) *Autobiographical Notes. A Centennial Edition* p.31. As quoted in D. Howard & J. Stachel (2000) *Einstein: The Formative Years, 1879-1909* p.1.

31. **the physics is inherent in mathematics:** Z. Artstein (2014) *Mathematics and the Real World* p.137-138.

32. **Beauty is the primary standard for scientific truth:** R. Augros & G. Stanciu (1984) *The New Story of Science* p.39.

33. **there is no permanent place in the world for ugly mathematics:** G. Hardy (1941) *A Mathematician's Apology* p.14.

34. **if they don't fit an elegant scheme, they are wrong:** M. Gell-Mann quoted in H. Judson (1980) *Search for Solutions* p.41.

35. **than to have them fit an experiment:** P. Dirac (1963) The evolution of the physicist's picture of nature. *Scientific American* **208**(5) p.45-53.

36. **to be taken as just that: a mathematical formalism:** R. Penrose (2004) *The Road to Reality* p.782.

37. **the theory should predict the results of measurements:** S. Hawking & R. Penrose (1996) *The Nature of Space and Time* p.121.

38. **the vast majority of student errors in math classes:** For example, see W. Bricken (1987) *Analyzing Errors in Elementary Mathematics.* Doctoral dissertation. Stanford University School of Education.

39. **links the abstract number to concrete notions of size, location and time:** S. Dehaene (2011) *The Number Sense: How the mind creates mathematics* p.246.

40. **geometry that already existed even in Babylonian times:** S. Wolfram (2002) *A New Kind of Science* p.792.

41. **not as justifications for mathematical rules:** A. Martínez (2006) *Negative Math* p.219.

42. **virtually nobody says that maybe it is the algebra that is defective:** Martínez, p.226.

43. **the associated subjectivities that permeate discourse — is crucial and nonnegotiable:** B. Rotman (1993) *Ad Infinitum* p.19.

44. **yet another mechanism of cultural imperialism:** The growth of big data has expanded this critique to include the impact of global quantification on our daily lives. "The bureaucratization of knowledge is above all an infinite excrescence of numbering." A. Badiou (2008) *Number and Numbers* §0.4

45. **eliminating the conceptual infrastructure of sets, functions and logic:** The root of almost all misinterpretations of boundary math is the attribution of concepts from conventional systems that do not exist within boundary systems.

46. **not essential to many overall properties of axiomatic systems:** Wolfram, p.1150.

Chapter 5

Structure

Everything should be made as simple as possible,
but not simpler.[1]
— Albert Einstein (1977)

This chapter is both a convenient summary of the James formal system collected into one place, and an advanced look at what is to come. It provides a succinct, structural description of the pattern rules that define James algebra. Chapter 6 next includes an introduction to the conceptual structure of these forms.

While learning, it is often beneficial to see in advance an overview of the upcoming content. This approach is called *advanced organization,* showing new concepts first so that when they are encountered later, in perhaps an unfamiliar context, the learner will not be taken aback by novelty. Cognitively, an advanced organizer allows the subconscious to process the new ideas and to lay a mental scaffolding that accommodates change. Our bodies do instantly understand the difference between inside and outside. Our skin is a very compelling (and built-in) model of a boundary. It's our cognition that needs reminding. I personally prefer the surprise of new ideas. You however might prefer to know what to expect.

boundary type	***James form***	☞	***interpretation***
round	(A)		$\#^{A}$
square	[A]		$\log_{\#} A$
angle	<A>		$-A$

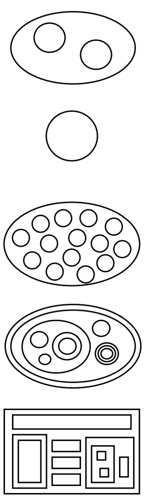
Figure 5-1: *James boundaries and an interpretation*

5.1 James Algebra

There are **three types** of James containers, represented in linear text by three different shapes of delimiting brackets. Figure 5-1 shows these bracket forms as well as a possible interpretation for conventional arithmetic. When empty, the three container types can be interpreted as the three fundamental concepts of arithmetic: 1, ∞ and 0. They can also be read as operations: power, logarithm, and inverse, with an *arbitrary* base, #.

Herein, James forms are presented from both the nested container perspective and from the numeric interpretation perspective. Neither perspective depends upon the other. The listings at the end of the chapter show permitted useful boundary transformations and connect the invariance that each supports to known expressions within conventional arithmetic and algebra.

some patterns of containment



To ameliorate the initial unfamiliarity, you may want to think of the notation for container forms as a foreign language. The goal is to see it as a clearer way to think about elementary arithmetic. We are also beginning at the very beginning, in conceptual territory so simple as to appear unfamiliar.

> In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premises because true consequences follow

from them, than for believing the consequences because they follow from the premises.[2]

Above, Whitehead and Russell are referring to deductive logic, but their comment applies just as well to an algebraic system. This chapter includes the entire content of James algebra, so that we may see all together the algebraic axioms and the structural consequences that follow.

Patterns and Principles

Many of the **concepts** of James algebra are succinctly listed on the Concepts page that follows this section. This conceptual structure rests upon a body of **general principles** related to boundary mathematics.

- The single underlying concept is *distinction*.
- Mathematics is the *experience* of abstraction.
- Experience is not a recording. Representation is not a reality.
- To participate in abstraction is to partition space, to construct a boundary.
- Boundaries both separate and connect.
- Representation and meaning are different sides of the same boundary.

our body is our interface

In James algebra structurally different forms of containment are defined to be equal by pattern equations. **Forms** are configurations of boundaries. The structure of a form can be changed only by following specified pattern rules. **Axioms** are transformation rules that are permitted as design decisions. **Theorems** are convenient transformations that follow directly from the axioms. **Frames** are notational structures that allow a conceptual organization of types of forms. **Maps**, or **interpretations**, are ways to convert James forms into conventional expressions within the arithmetic and algebra of numbers. Some James forms can be interpreted as imaginary numbers, some can be interpreted as being non-numeric.

Arithmetic

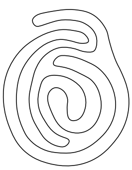
Here are the iconic Principles that define elementary arithmetic:

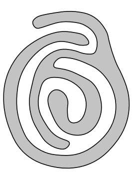
- **Existence:** something is different from nothing.
- **Accumulation:** parts do not condense.
- **Additive:** a sum looks like its parts.
- **Multiplicative:** each part of one touches each part of the other.
- **Hume's:** equality is one-to-one correspondence.

Axiomatic Style

A mathematical system must include some undefined ideas from which other formalized ideas are constructed. Whitehead and Russell:

> It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems simpler and easier.[3]





We begin then as did Spencer Brown, with one concept, that of **distinction**.[4] We represent the distinction by a boundary with a clearly delineated inside and outside. If you like, we begin by assuming the Jordan curve theorem as perceptually obvious.

Axioms are structural starting points, the first ground. There is an unlimited variety of axiom sets. The few interesting ones are those that provide some sort of power: more elegant concepts, greater understanding, learnability, philosophical appeal, perceptual obviousness, or importantly, a clear map to well known formal structures like logic, numbers, and sets. You might notice that there are very few **definitions**. From the structural perspective, containers have no inherent meaning other than their ability to contain. Definitions are abbreviations. Again, Whitehead and Russell:

> ..."definition" does not appear among our primitive ideas, because the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences.[5]

Meaning is off-loaded onto an interpretation, so that we may read containment structures as physical forms, as collections of nested boxes. All that is required is the algebraic tool of an equal sign and the ability to substitute equals for equals. The interpretation is dragged along with the valid transformations. If we must, we can assume that the definition of "=" is *is-confused-with*.[6] In Chapter 7 we will identify equality as *permitted structural transformation*. This is in contrast to the usual interpretation of = as invariance of numeric value.

Notation

Our notational zoo includes only four types of creature.

- **containers** represented by delimiting brackets, with empty containers serving as constants.
- **variable letters** that stand in place of an arbitrary container with arbitrary contents.
- the **equal sign** which identifies both identities and permissible pattern substitutions. Its twin, the **not-equal sign** identifies perceptually obvious difference.[7] The double-arrow ⇔ identifies equality between entire equations.
- various **abbreviations** and **meta-symbols** that stand in place of arrangements that otherwise would be awkward to represent. The two types include the meta-concepts ⇒, ⇔, *indeterminate*, and *void*, and the finite list abbreviations ... and ..N..

zoo

( ) [ ] < >

A a B b

= ≠

⇒ ⇔

...

..N..

void

bracket	name	use	chapters
		GENERAL	
{ }	`set delimiter`	conventional sets	
〔 〕	`shell`	value-neutral outermost	2,6-10,14
〈 〉	`logic boundary`	logic, not numeric	10,15
		UNIT-ENSEMBLES, DEPTH-VALUE	
( )	`parens`	depth-value group	3-4
< >	`angle`	negative ensemble	2
⟦ ⟧	`double-struck shell`	substitution operator	2-4,14
⦅ ⦆	`double-struck round`	depth-value base	11-12
		JAMES ALGEBRA	
o, ( )	`round`	numeric, exponential	6-13
[ ]	`square`	non-numeric, logarithmic	6-13
< >	`angle`	reflection, inverse	6,10-13

Figure 5-2: *List of typographic delimiters*

5.2 Remarks

Figure 5-2 lists all of the typographic brackets used in this volume. They fall into three distinct categories. The general delimiters are not formally part of James algebra; they are in the metalanguage. Depth-value delimiters are described in Chapters 2 through 4. The bracket system used in the rest of the volume is limited to the three James boundary forms: *round*, *square*, and *angle*. The empty round bracket has two representations, `o` and `( )`, for typographic convenience. To reiterate, the "shape" names are arbitrary and have no connection to geometry.

The overall **motivation** is to learn a new and *quite different* way of thinking and to apply that thinking to elementary arithmetic. What we discover along the way is that conventional arithmetic appears to be an accumulation of design decisions that, taken as a whole, lack conceptual coherence. Over thousands of years, we have stumbled our way into an arithmetic that works, but like all evolutionary processes, the assembly of parts is rife with unnecessary and redundant appendages. We teach this conceptual jumble to our children and as a consequence they too continue to stumble through elementary arithmetic, most leaving school loathing mathematics.

The three pages that follow the Concepts page show all of the pattern transformations and interpretations included in this volume: the axioms, theorems, frames, and maps. The two final pages show the transformation patterns introduced in Volume II and Volume III. To take the bull by the horns, how on Earth can any human wade through the apparently cryptic representations that follow? By learning to identify some simple patterns.

Endnotes

1. **opening quote:** A. Einstein (1977) *Reader's Digest* October 1977.

2. **because they follow from the premises:** A. Whitehead & B. Russell (1910) *Principia Mathematica* Preface p.v.

3. **to choose the one which seems simpler and easier:** Whitehead & Russell, p.91.

4. **with one concept, that of distinction:** Spencer Brown, *Laws of Form*, p.1.

5. **but are, strictly speaking, mere typographical conveniences:** Whitehead & Russell, p.11.

6. **the definition of "=" is is-confused-with:** This perceptual perspective follows Spencer Brown's informal definition of the equal sign, p.69.

7. **the not-equal sign identifies perceptually obvious difference:** The asymmetry between = and ≠ is surprising. If two forms are not equal, we must be able to see the difference. If two forms are equal, it may not be immediately apparent since they may look different. Axioms, then, identify forms for which we cannot trust our perceptions directly. Axioms are designed confusion. The equal-sign unifies what we might otherwise believe to be different.

Concepts

Void

Void has no properties. (Nothing is not something.)
Form is either not nothing or an illusion.
Void-equivalent forms may vary in structure but not in relevance.
Void-equivalent forms are syntactically inert and semantically irrelevant.

Containers

Containers represent distinctions.
Everything is a container.
There is only one relation, `contains`.
Empty containers are units.
Containers are both object and process.

Structure

Forms are patterns of containment.
Valid forms can be constructed physically.
Forms can be represented in many multi-dimensional notations.
Containers support nesting and not sequence.
Containers are not limited to a specific capacity (no arity).
The contents of any container are mutually independent.

Axioms

Axioms subdivide existent forms into discrete groups.
Axioms define the forms that are void-equivalent.
All canonical forms are unequal.

Equality

Containers with equal contents are equal.
Equals can be substituted for equals.
Removing identical outer boundaries maintains equality.
Removing equal contents maintains equality.
Equality is quantized dynamically by transformation steps.
Forms change meaning only when they cross a boundary.

Arithmetic

To count is to identify, categorize, indicate, fuse and label.
Addition is putting forms into the same container.
Multiplication is putting square forms into a round container.
Exponential and logarithmic bases are defined by the interpretation.
Inverses are represented by the same boundary in different contexts.

Axioms

Arithmetic

		page
( ) ≠ *void*	**existence**	168
( ) ( ) ≠ ( )	**unit accumulation**	170
([ ]) = [( )] = *void*	**void inversion**	184
<( )> ( ) = *void*	**unit reflection**	46

Algebra

([A]) = [(A)] = A	**inversion** *enfold ⇄ clarify*	184
(A [B C]) = (A [B]) (A [C])	**arrangement** *collect ⇄ disperse*	193
A <A> = *void*	**reflection** *create ⇄ cancel*	241

Theorems

Frame

(A [ ]) = *void*	**dominion** *emit ⇄ absorb*	242
([A][o]) = A	**indication** *unmark ⇄ mark*	218
([A][o..$_N$..o]) = A..$_N$..A	**replication** *replicate ⇄ tally*	219

Reflect

<<A>> = A	**involution** *wrap ⇄ unwrap*	241
<A><B> = <A B>	**separation** *split ⇄ join*	241
<A <B>> = <A> B	**reaction**	242
(A [<B>]) = <(A [B])>	**promotion**	244
(A <[<B>]>) = <(A <[B]>)>	**promotion** *demote ⇄ promote*	

Frames

([ ]) = *void*	**void**
([A]) = A	**inversion**
(A [ ]) = *void*	**dominion**
(A [B C]) = (A [B]) (A [C])	**arrangement**
([A][o]) = A	**indication**
([A][N]) = A. .$_{N}$. .A	**cardinality**
(A [B])	**magnitude**
(o [N])	**unit magnitude**
(<o>[N])	**decimal**
(J [A]) = <A>	**J-conversion**

Ensembles

⦗a|b|c⦘ = ⦗a b c⦘	**fuse**
⟦A • E⟧ = ⟦E • A⟧	**commute**
A <A> = *void*	**reflect**

Depth-value

•. .$_{N}$. .• = (•)	**group**
(a)(b) = (a b)	**merge**
⦅N⦆ = ([N] o)	**depth-value**

Logic

Arithmetic

⟨ ⟩⟨ ⟩ = ⟨ ⟩	**calling**
⟨⟨ ⟩⟩ = *void*	**crossing**

Algebra

⟨A ⟨ ⟩⟩ = *void*	**dominion**
⟨⟨A⟩⟩ = A	**involution**
A ⟨A B⟩ = A ⟨B⟩	**pervasion**

Maps

expression ☞ ***James form***

UNITS

1	( )
0	< >
–1	<( )>
–∞	[ ]
$\log_{\#} -1$	[<( )>]

INVERSE

A	A
–A	<A>
1 ÷ A	(<[A]>)

ARITHMETIC

A + B	A B
A – B	A <B>
A x B	([A] [B])
A ÷ B	([A]<[B]>)

BASE

B^{A}	(([[B]] [A]))
B^{-A}	(([[B]] [<A>]))
$B^{1/A}$	(([[B]] <[A]>))
$\log_{B} A$	(<[[B]]>[[A]])

EMBEDDED BASE

#	(o)
$\#^{A}$	(A)
$\log_{\#} A$	[A]

PARALLEL

counting	1 + . . . + 1	o . . . o
addition	A + . . . + Z	A . . . Z
multiplication	A x . . . x Z	([A]. . .[Z])
fraction	(A x. . .x M)÷(N x. . .x Z)	([A]. . .[M]<[N]. . .[Z]>)

Volume II

Equations

A = B ⇔ A <B> = *void* = <A> B	**reflection bridge**
A = B ⇔ {A} = {B}	**compose context**
{A} = {B} ⇔ {A C} = {B C} C ≠ [ ]	**compose content** *decompose ⇄ compose*
(A) = B ⇔ A = [B]	**equality inversion** *cover ⇄ cover*
A C = B ⇔ A = B <C> C ≠ [ ]	**equality reflection** *move ⇄ move*

Two Boundary

<A> = (⟦A⟧)	**two-boundary angle**
[A] = ⟦(⟦A⟧)⟧	**two-boundary square**

Volume III

Non-numeric

AXIOMS

[ ] [ ] ⇒ [ ]	**unify**
<[ ]> <[ ]> ⇒ <[ ]>	**unify** → *unify*
[ ] <[ ]> ⇒ *indeterminate*	**indeterminacy**

THEOREMS

(A <[ ]>) = (<[ ]>)	**dominion II** *emit ⇄ absorb*
(A [[ ]]) = [ ]	**dominion III**
[[ ]] = J <[ ]>	**double-square**
[[[ ]]] = <[ ]>	**triple-square**

INTERPRETATIVE AXIOM and THEOREMS

(<[ ]>) = <[ ]> = [<[ ]>]	**infinite interpretation**
<(<[[ ]]>)> ≠ *void*	**infinitesimal**

Volume III

J Patterns

Theorems

J = [<o>]	**definition of J**
<A> = (J [A])	**J-conversion**
J J = *void*	**J-void object**
[<(J)>] = *void*	**J-void process**
([J][2]) = *void*	**J-void tally**
J = <J>	**J-self-inverse**
[<(A)>] = A J	**J-transparency**
A (J [A]) = *void*	**J-occlusion**
J (J [J]) = *void*	**J-self-occlusion**
J = <[A]>[<A>]	**J-invariant**
[<J>] = J [J]	**J-absorption**
<(J/2)> = (<J/2>)	**J/2-toggle**

Parity

N *even*	([J][N]) = ([J][<N>]) = *void*	**J-parity**
N *odd*	([J][N]) = ([J][<N>]) = J	
N *even*	J ([J][N]) = J	**J-parity**
N *odd*	J ([J][N]) = *void*	
N *even*	J ([J]<[N]>) = ([J]<[N]>)	**J-fractions**
N *odd*	J ([J]<[N]>) = *void*	

J Frames

(J [A]) = <A>	**J-conversion**
(J [<A>]) = A	**J-involution**
(J <[A]>) = <(<[A]>)>	**J-angle**
(J [J]) = J	**J-self**

Complex

i	☞	(J/2 [o]) = (J/2)	*form of* i
π	☞	(J/2 [J])	*form of* π
a + bi	☞	a (J/2 [b])	*form of complex number*

Chapter 6

Perspective

Notation...is extremely important in mathematics.
A seemingly modest change of notation may suggest
a radical shift in viewpoint.[1]
— Barry Mazur (2003)

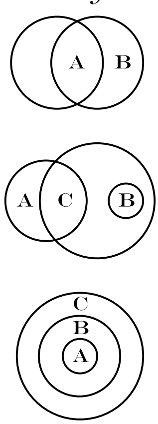
James algebra is named after Jeffrey James. He and I developed the algebra as his 1993 Master's Thesis at University of Washington, *A Calculus of Number Based on Spatial Forms*. This work has not been published previously. Most of the results in Chapter 7 through Chapter 12 are included in Jeff's thesis. James numbers take their inspiration from Charles Sanders Peirce, who introduced boundary logic at the turn of the 20th century; from *Laws of Form*, the seminal work of the late George Spencer Brown; and from the work of Professor Louis Kauffman at the University of Illinois at Chicago.

James algebra is a radical reconceptualization of how we represent and think about conventional numeric operations `{+, –, x, ÷, ^, √, log}`. Like the unit-ensembles described in Chapter 2, James forms are *additive*. Forms are added together by putting them together into the same container.[2] Unlike unit-ensembles, multiplication is represented by a specific configuration of boundaries, rather than as the substitution operation. The advantage of this approach, especially during transformation, is that multiplication can be treated as a static pattern. By revoking the commutativity of substitution, we return match-and-substitute to its position as the only method of transformation, a generic mechanism that converts axioms into theorems and tools.

The axioms of modern algebra evolved in tandem with linear typography and with sequential (causal) thinking. The axiomatic structure of algebra and set theory do not capture the *essence* of what numbers are nor how they work since neither system rests upon the Additive Principle. We will presume that the manipulative use of numbers throughout history and the modern symbolic perspective on numbers both refer to the same numbers. Different constructions of the same concept provide alternative perspectives on that concept, perspectives that can enrich and generalize our current understanding. Our conventional perspective on arithmetic, the one currently taught in schools, is extremely useful for business and for scientific professionals, quite useful for sequential computers, and definitely a nuisance for educators, for students, for a great majority of Americans, and for parallel, concurrent and distributed computation.

6.1 Diagrammatic Math

Euler diagrams



Euler was first to propose a diagrammatic method of logic. His **Euler diagrams** associate embedded and overlapping circles with logic syllogisms. He explains:

> The foundation of all these forms is reduced to two principles, respecting the nature of *containing* and *contained*. I. Whatever is in the thing contained must likewise be in the thing containing, and II. Whatever is out of the containing must likewise be out of the contained.[3]

During the nineteenth century, non-Euclidean geometries were discovered and formalized. The previous two thousand years had established Euclid's geometry as the sacrosanct definition of mathematical rigor.[4] But Euclid's parallel line postulate was too narrow, it worked only for flat surfaces. There are geometries, such as the surface of a sphere, for which the parallel postulate does not hold. In the nineteenth century, this trauma of discovery shook the mathematical world so fundamentally that the ancient Greek perspective of deriving mathematical knowledge from diagrams was completely abandoned, in favor of purely symbolic approaches. Herbert Simon: "Rigor, it was believed, called for reasoning to be formalized in symbols arranged in sentences and equations."[5]

Euclid
circa 450–350 BCE

Hilbert's program at the turn of the twentieth century set out to express mathematical reasoning in finite strings of symbols. Mathematical diagrams and other sensory/experiential forms were widely purged from rigorous mathematics. In particular Euler diagrams, Venn diagrams, Frege's deduction trees and C. S. Peirce's existential graphs, each of which has been shown to be sound, were all suppressed and largely forgotten.

Leonhard Euler
1707–1783

John Venn
1834–1923

Charles Sanders Peirce
1839–1914

Gottlob Frege
1848–1925

The **syntax/semantics barrier** is deeply implicated in the migration to linear structure. The meaning of words and strings of symbols became entirely separate from the words and symbols themselves. Understanding was buried underneath arbitrary obscurity. Symbols require augmentation, meaning must be added separately. Here's Larkin and Simon:

> The fundamental difference between our diagrammatic and sentential representations is that the diagrammatic representation preserves exactly the information about the topological and geometric relations among the components of the problem, while the sentential representation does not.[6]

However, the Participation Principle reminds us,

Strings of symbols do impact meaning
by limiting how we think about what we are describing.

Bertrand Russell
1872–1970

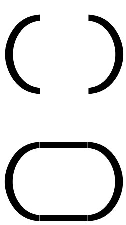
> There is, however, a complication about language as a method of representing a system, namely that words which mean relations are not themselves relations, but just as substantial or unsubstantial as other words. In this respect a map, for instance, is superior to language, since ... a relation is represented by a relation.

Bertrand Russell continues,

> I believe that this simple fact is at the bottom of the hopeless muddle which has prevailed in all schools of philosophy as to the nature of relations.[7]

James algebra builds all structure out of *icons, images and diagrams*, out of containers that support the visual relation of inside and outside. Logicians Barwise and Etchemendy:

> Diagrams are physical situations. They must be, since we can see them.... By choosing a representational scheme appropriately, so that the constraints on the diagrams have a good match with the constraints on the described situation, the diagram can generate a lot of information that the user never need infer. Rather, the user can simply read off the facts from the diagram as needed. This situation is in stark contrast to sentential inference, where even the most trivial consequence needs to be inferred explicitly.[8]



James algebra does not embrace linear, typographic communication. *James forms are spatial.* They are iconic rather than symbolic. The typographic representation used herein shows containers as string delimiters such as ( ), with the image of the container broken into a left and a right half. However, the sequencing and the fracturing of boundaries is an *accident* of the way that our typewriters and our textual languages are constructed, and a potentially confusing distortion of the image of a container.[9]

Our theme is that numbers are sensory, diagrammatic, experiential. We do not need to obscure how arithmetic works with veils of symbols and tables of relations. Hiding meaning behind memorized convention both limits and distorts thought. Alfred North Whitehead: "By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems."[10]

Boundary Algebra

First and foremost, James forms are **void-based**. *Void* is intended to look like what it means. It literally does not exist. In text the label *void* is a something, it's a word. Words are not nothing. They support reference and eventually communication. In contrast, nothing is not something, it does not support reference. *Void* is in our shared descriptive metalanguage, however it is not part of the James system.

Void-equivalent structures and patterns are irrelevant to meaning. These forms exist solely in notation. Void-equivalent forms are illusions that arise from an empty page. Void-equivalence implies that

— *Absence* has no properties.

— Containers are *permeated* with void forms.

— *Void-equivalent forms* are background potential. They can be freely created and deleted.

— *Empty* containers can be seen as units.

— Transformations *create and delete* structure.

James forms are **containers** with these properties

— Forms are *patterns* of containment.

— Forms are patterns of *physical* containers.

— A container is an *object* from the outside and a *process* from the inside.

— Contents are *mutually independent.*

— Concepts are *networks* of `contains` relations.

formal arithmetic is putting stuff into the right boxes

A **calculus** consists of a notation for representing objects, a collection of permitted transformations and a collection of basic facts. We are representing physical containers by delimiting boundaries and basic facts by empty containers. Common use of numbers and arithmetic can be seen as putting things into containers and rearranging those containment relations by following the pattern rules defined by the three James structural axioms.

The physicality of containers means that we can viscerally interact with James forms. We can elect, for instance, to build James forms out of physical objects such as blocks, or out of physical enclosures such as rooms. The James axioms define the coordinated behavior of various patterns of containment. A sequence of structural transformations can be animated. The creation, deletion and rearrangement acts that constitute both proof and computation can be presented in videos as dynamic animations. Many transformations can happen at the same time since (other than containment) each container is independent of the others.

The inside of a container supports concurrent transformation of its contents, just like the inside of a theater full of people supports concurrent breathing. In that metaphor, all transactions are between a person and the air in the room, between content and context. There is no interaction between contents, no direct connections between people in the room. All may be immersed in (contained by) watching the movie. None are watching the other movie-goers breathe.

In James algebra, there are no instances of counting, ordering or grouping because there is no imposition of structure other than that of containment. Importantly, only one axiom permits rearrangement of structure, the forte of string languages. The other two axioms (and most of the theorems) are void-based, they eliminate structure by erasure/deletion, by casting structure into *void*.

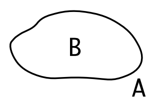
boundary	***unit***	***interpretation***	***operator***	***interpretation***
angle	< >	0	<A>	−A
round	( ), o	1	(A)	$\#^{A}$
square	[ ]	$-\infty$	[A]	$\log_{\#} A$

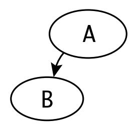
Figure 6-1: *James units and operations*

6.2 Container Types

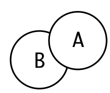
James algebra uses three distinct types of containers to express numeric and non-numeric structure. Figure 6-1 shows the round boundary, the square boundary, and the angle boundary. In this volume, we'll stick to the interpretation of James forms in Figure 6-1.[11]

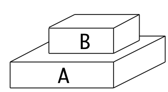
Boundary forms are icons. Pictorial forms trigger not only different conceptual models, they trigger different physiological processes. Transcription is therefore more than a cognitive shift, it implicates different perceptual systems and a different behavioral vocabulary.

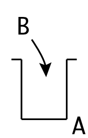




The only relation within a boundary calculus is that of **containment**, a *minimal conceptual basis* consisting of one binary relation. The `contains` relation is quite general. When expressed within logic, containment can be interpreted as `implies`. When expressed as an acyclic network, containment is `directly-connected-to`. When expressed as a set, it's called `is-a-member`. When expressed as a number, it is `successor`. When expressed as a map, it's `shares-a-common-border`. Within the context of a pile of blocks, `contains` becomes `supported-by`. When seen as a family relationship, it is `parent-of`. When described as an abstract mathematical structure, it is a *rooted tree*. All of these metaphors share a collection of common characteristics that are concretized by the properties of physical containers. The fundamental *concept* underlying containment is **distinction**: a container distinguishes inside from outside.







`([A]) = [(A)] = A`	**inversion** *enfold ⇄ clarify*
`(A [B C]) = (A [B]) (A [C])`	**arrangement** *collect ⇄ disperse*
`A <A>` = *void*	**reflection** *create ⇄ cancel*

Figure 6-2: *Pattern axioms of James algebra*

Volume II looks at the structure that the binary relations `contains`, `implies`, `is-a-member`, `successor`, and `parent-of` have in common: each makes a distinction between container and contained. This chapter provides an initial discussion of the mechanisms of the James pattern algebra of distinctions.

Pattern Axioms

Figure 6-2 shows the pattern axioms of James algebra, the transformations that are designed to define the behavior of the arithmetic and algebra of numbers.

Calculi based on an equal sign are called *algebras*. The algebraic style of boundary math includes maintaining equality by transforming containment relations that match clearly defined patterns. If a containment pattern does not match a rule, then it cannot be changed. More generally the **Axiomatic Principle** constrains what can be done within a formal system.

Axiomatic Principle

If it is not explicitly allowed, then it is forbidden.

Two of the three James axioms define the interplay between round and square boundaries, while the third defines the behavior of angle boundaries. It is convenient to be able to identify the direction of application of each axiom when describing computational steps. Figure 6-2 also provides these names. We will later develop several convenience theorems. The constructive demonstration of these theorems never strays far from the three simple axioms. All transformations of containment patterns are essentially simple, there are no particularly subtle or creative theorems in this volume.

The most important characteristic of these axioms is that two of them specify how to delete structure. Both implicate only one form (labeled A), so that they both require only simple pattern-matching. Remarkably, this leaves all of the complexity of numeric algebra isolated in one pattern transformation.

In general, a **variable** within an algebra stands in place of an *arbitrary* form. In James algebra, this idea is slightly more complex. Since the algebra is void-based, a variable might stand in place of nothing. Variables that are not void-equivalent stand in place of a single container and its contents. It is a violation of the structure of containment to have multiple forms standing in the same space without an outermost container. However, it is often typographically convenient to leave the outermost container implicit. We can display, for example, either (oooo) or ([oooo]) or oooo, with the understanding that the outermost container of oooo may be unwritten but it is certainly present.

explicit
([A B C])

implicit
A B C

arithmetic
([ ][ ])

Deleting variables exposes the arithmetic structure of a form. The algebra of James forms is thus strongly connected to its arithmetic.

algebra
([A][B])

Partial Ordering

Forms are specific patterns of containment. All the possible containment patterns constitute the **language** of James forms. The mathematical abstraction that comes closest to describing James forms is a partial ordering.

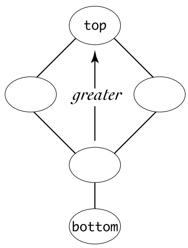
A **partial ordering** is a graph consisting of nodes and links. The nodes are containers that are delineated by a boundary. The links are containment relations. A physical, or finite, strict partial ordering has these defining characteristics:

— There is a *top* node and a *bottom* node.

— There is a *direction*, every node is on a path between top and bottom.

— Links identify specific directional relations between nodes.

— Nodes bound links.



Within the theory of relations, a partial ordering has three characteristics

— **irreflexive:** no node is linked to itself

— **antisymmetric:** no node is above or below itself

— **transitive:** you can travel to distant nodes

One objective of Volume II is to look at these rather strange notions in detail. In general they fail to convey the intent of a containment pattern. Consider `contains` to be `parent-of`. You cannot be your own parent (irreflexive), and you can't be your parent's parent (antisymmetric). As well, your parents are not the `parent-of` your children (intransitive). Technically then containment is not a partial ordering because it is not transitive. There is a transitive concept that we might call `contained-at-any-depth`. In our example it would be the `ancestor` relation. But the deeper point is that using conventional abstractions based on sets does not particularly help us to think differently about the formal structure of distinction.

Interpretation

Figure 6-3 provides a quick introduction to the **interpretation** of boundary configurations that we will use. These patterns unify counting, adding, multiplying,

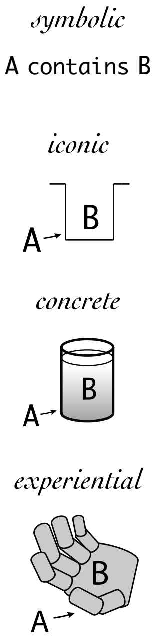
expression	☞	*form*
	INVERSE	
A		A
−A		<A>
1 / A		(<[A]>)
	ARITHMETIC	
A + B		A B
A − B		A <B>
A x B		([A] [B])
A / B		([A]<[B]>)
	BASE	
B^{A}		(([[B]] [A]))
B^{-A}		(([[B]] [<A>]))
$B^{1/A}$		(([[B]] <[A]>))
$\log_{B} A$		(<[[B]]>[[A]])
	EMBEDDED BASE	
#		(o)
$\#^{A}$		(A)
$\log_{\#} A$		[A]

Figure 6-3: *Algebraic operations to patterns of containment*

raising to a power, and the assortment of inverse operations. However, numbers and numeric operations are but one interpretation of what a container form might mean. When translating from one language to another (here, for example, between configurations of containers and conventional arithmetic), the more primitive, less redundant and therefore *foundational* language will have multiple alternative interpretations within the more sophisticated and complex language. Thus, the single boundary configuration A <B> can be read both as A + −B and as A − +B. Interpretation from a simpler foundation is one-to-many.

6.3 Multidimensional Form

A primary reason for going to all the trouble to learn this new sensual language is to learn new ways of thinking. It is not the concepts represented by the James language that are multi-dimensional, it is the language itself that has different dialects or “notations” expressed in different dimensions. One implication is that a series of transformations can be *animated*. Another implication is that many transformations can occur concurrently, all at the same time.[12]

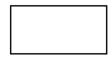


When you stop to consider the rationality of symbolic representation, it becomes clear that symbols are highly discriminatory against our physical evolutionary heritage. The vast majority of the neurons in our brains are dedicated to managing the interface of our physical body with physical reality. Everybody lives in a body, only a very few of us live in the conceptual fantasy of the Platonic reality associated with mathematics.[13] Abstraction is of interest to only a small portion of a brain; the skills of abstraction are exceedingly difficult to teach. The symbolic math currently taught in schools expects us to abandon both sensation and experience in favor of unnatural cognitive acts. No wonder students find it difficult to learn this disembodied language.

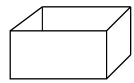
Boundary languages are *visceral*. Interpretation will remain constant as the boundary representation is transcribed across dimensions, from 1D strings to 2D icons to 3D architectures to 4D temporal experiences. There is no abstract/concrete dichotomy, so that boundary languages are much easier to understand. No mind/body split, so boundary forms are much easier to tolerate. In contrast, string encoding cannot be experienced, it must be learned via memorization. Consequently string languages remain necessarily *cerebral*. **Mathematical nominalism** holds that mathematics is about objects that exist. Container languages provide nominalistic consistency by requiring that formal concepts too have a manifest form.

It is a distinct advantage to represent mathematical concepts across many different spatial formats, not only symbolically but also diagrammatically, physically and experientially. The printed page limits representation to symbolic and iconic forms, but by projecting volumetric forms onto paper, we can approximate concrete and experiential languages. The image of a box can elicit imagination of a box. Leibniz: "The best signs are images; and words, insofar as they are adequate, should represent images accurately."[14]

2D



projection of 3D onto 2D



Dialects

Containment relations themselves can be expressed not only as configurations of containers, but also as maps, networks and symbolic equations. Most string languages can also be expressed as spatial networks. A difference, though, is that a James icon embodies its operational semantics. In effect there is no distinction between form and intent. The container boundary is the only diagrammatic component we will need. It visually and computationally preserves the dependency of containment, which itself can be interpreted as nesting, sequence, stacking, connectivity and several other types of *physical* relationship between container and contained, as illustrated in Figures 6-4 and 6-5. Containers provide a built-in visualization of dependency, appealing for both form and interpretation to our hands and our eyes, rather than to our ears and vocal cords.

Figure 6-4 shows the James form of multiplication expressed as one-, two-, and three-dimensional containers.

- The **string** dialect is digital and encodable. The language consists of delimiters in fractured bracketing relationships with one another.
- The **bounded** dialect shows two-dimensional containment. The language consists of enclosures in nesting relationships with one another.

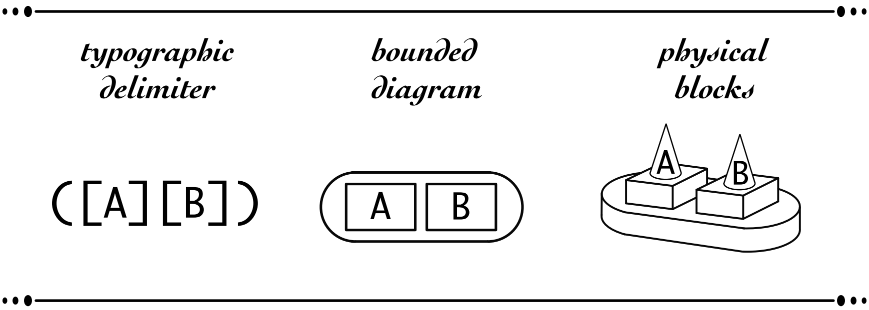


Figure 6-4: *One-, two- and three-dimensional forms of multiplication*

— The **block** dialect is manipulable. The language consists of physical objects in stacking relationships with one another.

Figure 6-5 shows James multiplication in some other spatial dialects of containment, including two-dimensional maps and paths, three-dimensional physical rooms and dimension-free networks.

— The **network** dialect is a traversable acyclic graph. The language consists of nodes and links.

— The **map** dialect is a traversable territory. The language consists of areas with shared borders.

— The **path** dialect shows border crossings that define the boundary form. The language consists of a single instance of each type of boundary, together with a directed path crossing the boundary archetypes.

— The **room** dialect is a three dimensional environment inhabited by a participant. The language consists of rooms and doors.

For examples of James *arithmetic* in each of these notations, delete the variables A and B in Figure 6-4 and Figure 6-5.

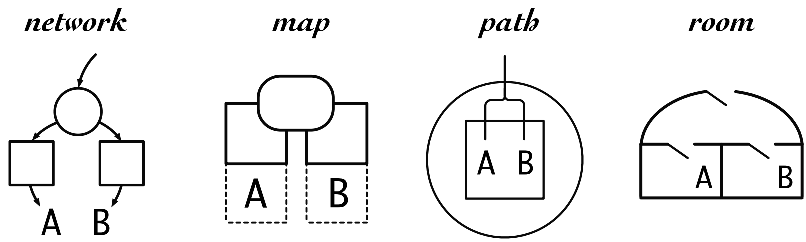


Figure 6-5: *More forms of multiplication*

Hybrid Notation

Hybrid notation mixes boundary structure with *special symbols* that stand in place of specific boundary forms. Usually these special symbols are reminiscent of conventional notations. Special symbols will be necessary in order to make the upcoming boundary forms of π, i, cosine, and the like easier to read. A hybrid notation makes unfamiliar forms a bit more readable. It is particularly handy when the same boundary pattern occurs multiple times within a form.

The archetypical special symbol is the variable A, which stands in place of any James form. *Naming* boundary structures is safe so long as unique names are assigned to unique patterns. And of course we can conveniently move back and forth between a name and the form that the name identifies. We can also engage in symbolic abbreviations for both typographic and reading convenience. We'll sometimes use natural numbers, for example, to stand in place of ensembles of units. It will be convenient to use the special symbol "5", for example, to stand in place of the form ooooo. Sometimes however it is conceptually important to *show* these ensembles explicitly.

o	☞	1
oo	☞	2
ooo	☞	3
oooo	☞	4

and so on

There are two abbreviations that we will be seeing a lot so they may as well be introduced now. The **hash mark**, #, indicates an *arbitrary base* for polynomial numbers and for exponents and logarithms. A new idea facilitated by James algebra is that exponents generally do not need a base for transformation to move forward. In an algebraic context, *which* number is being multiplied together many times (i.e. the *base*) often does not matter. Often an entire form is standardized for one base only. The base then is a global feature of the form rather than a local feature of each boundary, an example of the Communality Principle.

(A) ☞ $base^A$
[A] ☞ log_{base} A

J represents another new idea, a new imaginary number, the **logarithm of negative one**. The special symbol J stands in place of the boundary configuration [<( )>]. J is the subject of Volume III and makes only a few appearances until then.

Object/Process

How should we think about −3, or 1−3, or 1/3, or √3, or log 3? These representations signify exact numbers, yet somehow we have embedded the operations of subtraction, division, root and logarithm into their notations. Can we do no better than to describe some numbers as operations on other numbers? Can combinations of units and operators legitimately be called numbers? This question can be phrased in the simplest of terms:

Is 1/3 *a single number*
or is it two numbers combined by an operation?

Does 1/3 explicitly identify an actual number? Yes, in conventional terms, it is a member of the class of rational numbers. Do the rational numbers cover the number line? No, no more than the natural numbers cover the number line. Does 1/3 explicitly identify a ratio or comparison? Yes, in conventional terms a fraction is a ratio arising from the comparison of the magnitude of two natural numbers. Here we compare 1 to 3. 1/3 identifies the proportion of the ensemble ••• that is •. So in a conventional sense, 1/3 is both one and two numbers, depending upon our purpose.

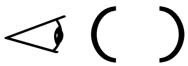
The use the same notation to represent both concept and process is an example of the Participation Principle that is widespread in mathematics. *What a number means depends upon how you look at it*. This dual usage is extremely handy, since it permits an expert in mathematical abstraction to arbitrarily shift the interpretation of an expression between object and operation. 3 can be seen both as a number and as an operation on numbers, depending upon what is most convenient at the time. 3 can represent a count, or it can represent an act of replication. Not only does 1/3 behave as both a number and a comparison, the transformation rules for operating upon the structure 1/3 can differ depending upon its intended interpretation.[15] That's why expertise is needed to use conventional notations.

We are also asking an educational question: is this flexibility helpful to those learning how to manipulate numbers? Is the object/operator duality an essential aspect of what we mean by *number*, or is the duality an artifact of one particular way of looking at numbers? Might the distinction between object and operator be illusion, perhaps imposed by abstractions that are more complicated than is necessary? Are students and their teachers aware of the entanglement of things and processes? Why do fractions traumatize so many young students?

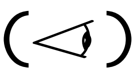
The object/operator duality of conventional numbers is reminiscent of the particle/wave duality of atomic particles. The founders of quantum mechanics became comfortable with the idea that an electron will behave both like a wave and like a particle, depending upon how we choose to observe it. What an electron *is* depends upon how we want it to appear. Similarly, what a number is depends upon what we want of it. Perhaps we should develop a conceptually simple arithmetic in which numbers do not take on the properties of operators. In which, for example, numbers add but do not multiply. Perhaps numbers can be simplified by limiting how they are permitted to behave.

James numbers provide yet another perspective on the relationship between object and operator. Rather than embedding operations into the structure of numeric forms, James algebra uses *viewing perspective,* a feature of container-based math, to distinguish between object and operation. When we look at the outside of a container, ignoring its contents, then the container appears to be a very simple **object**, a container. When we look at the contents inside a container, the container itself is a **process**, a very simple process, `contains`. Viewed from the outside, a James container can represent a number. Looking at the inside shows us how that number is constructed. This is very similar to our use of function notation, where `f(x)` stands in place of the intended result of the process `f` acting upon the object `x`. But `f(x)` shows an inconsequential component of the inner structure, the arbitrary label used to access the functionality identified by the label `f`. The difference is that the contents of James containers are the entire inner structure.

outside/object



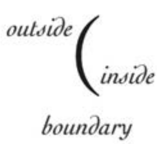
inside/process



6.4 Features

James numbers share these features with the other boundary systems:[16]

- There are significantly fewer abstract concepts.
- Everything is a containment relation.
- Void-equivalent forms can be ignored.
- Contents do not interact.
- Conventional operations condense into a few structural axioms.
- Proof and computation are achieved by pattern substitution and deletion.
- The algebraic theory of groups is not relevant.



Here is a slight expansion of each of these ideas within the context of James algebra.

Significantly fewer concepts

Perhaps the most challenging new idea is that many of the conventional *concepts* within arithmetic and algebra are unnecessary in order to conduct the conventional operations of arithmetic and algebra. Arithmetic can be far simpler than what we have been taught. The essential concepts of James algebra do *not* include zero, commutativity, associativity, arity, one-step-at-a-time processes, and symbolic strings. Instead there is a deep visceral unity across counting, addition, multiplication, power, subtraction, division, roots and logarithms.

Containment relations only

James algebra is based on *containment relations*. What is inside a specific container tells us what that container acts upon, or `contains`. Different concepts are expressed by three different containment relations, which we have represented as `( )`, `[ ]` and < >. A numeric interpretation is not necessary and is not embodied within the axioms of the algebra.

Semantic use of *void*

The absence of a container has meaning. We can interpret absence as numeric zero. An empty container is a *unit*. Emptiness creates unity. Unity is absence of interior parts. The interior of every container, even those with contents, is pervaded by *void*. Void-equivalent forms are everywhere and in unlimited supply. *Bounded space* cannot and does not have properties, unlike the sequencing properties of the spaces between these words and the spaces between the characters of each word. *Void* is not synonymous with emptiness, it is more like the physical space that is underneath all physical objects. This leads to the fundamental **Principle of Void-equivalent Forms** (and of the *void* itself).

Void Equivalence

Void-equivalent forms are syntactically inert
and semantically irrelevant.

In the numeric domain, void-equivalent forms are both blind to sign and blind to multiplicity.[17]

Independence of contents

There are no direct relations between contents. The only relation is between a container and each of its content forms. Another way to say this is that the void shared by contents of the same container has no properties and thus cannot support relations between contents. This explicit independence permits the implementation of extensive parallelism within forms. Concurrency is a native mode of thought, analogous to sight rather than to speech.

Algebraic operations condense into a few patterns

Three generic transformations on patterns of containment are taken as axioms. An interpretation of round-brackets as powers and square-brackets as logarithms permits ease of mapping James forms to conventional arithmetic. When nested, the two types of boundary maintain alternating exponential and logarithmic contexts which permit smooth transition between addition and multiplication.

Proof by pattern-matching and substitution

Axioms *statically* define patterns that are equivalent, and *dynamically* permit transformation between patterns. Structural axioms are implemented by matching a given structure to a permitted pattern and then replacing it by a certified equivalent structure. Two axioms identify void-equivalent forms, permitting deletion of structure. The third permits rearrangement. This makes computation and verification short and elegant.

Absence of the laws of algebra

Commutativity and associativity are interpreted as sequential concepts that are not relevant to a spatial, parallel form of arithmetic.[18] There is no ordering imposed on the contents of a container. Grouping is defined solely by containment. Unlike functions which specify a precise number of arguments, a container can accommodate any number of contents (arguments). There is no concept of *arity*. The `0` null object of addition and the `1` null object of multiplication are opposite sides of the same distinction, `1` from the outside and `0` from the inside.

Incidentally, if you are counting, we have eliminated three of the five fundamental properties of algebraic groups. From the James perspective only inverse and distribution are fundamental.[19]

6.5 Strategy

James algebra may not be initially friendly to the eye, primarily due to its unfamiliarity. It becomes friendly to the mind only after exerting some effort to achieve familiarity. Familiarity can be achieved cognitively or iconicly or physically. Do not expect to easily recognize a radical revision of the mathematics that has been standardized across the globe for a century. Do not expect to easily shed the perspective embodied within twelve years of grade school and high school mathematics courses. Or, if you choose, simply adopt the mind of a novice, someone who has never encountered a number greater than two.

Mathematical concepts themselves are defined by containment patterns. We do not have to abandon the familiar concepts of symbolic math, since carrying them into a reading of a James form at least provides comfort in familiarity. The adventure though is in finding other concepts, *iconic concepts*, overlaid and interacting with our familiar concept of number. It is these new perspectives that provide the motivation to explore. Could it be that the categories of numbers that we hold to be important (naturals, integers, rationals, irrationals, transcendentals, reals, imaginaries) have accreted over time without *common varieties of numbers*

0
1
43
−5
½
5/13
32/5
2 1/7
2.97
10110
√2
log 7
e
π
i
3+4i

fundamental organization? Might there not be an alternative classification of number that makes more sense? Similar to Conway numbers[20] (aka surreal numbers), this exploration is not yet pragmatic; the goal is simply to shed some new light on what it means to be a number. A deeper objective is to show that symbolic arithmetic as we know it is not fundamental, but rather just one of many formal approaches to magnitude.

The exposition of James numbers includes both definition and transformation, as well as guidance about how to use the new ideas. Some of the new iconic concepts are not about numbers at all. They are spatial rather than numeric. These same concepts also permit us to describe and to simplify conventional *logic*, thus achieving a unification of number and logic, of algebra and proof.[21] One small change in the rule that governs units can change arithmetic into logic. Visually:

natural numbers	( ) ( ) ≠ ( )
elementary logic	( ) ( ) = ( )

Let's first look at James numbers dispassionately, as just another way to think about magnitude. We will establish a map between James forms and conventional numbers. Later we will ask: if both systems describe the same thing, why do they appear to express such different concepts? How can the single idea of containment cover the diversity of ideas embodied in conventional arithmetic? More specifically, is the (bewildering to the uninitiated) array of number types in the sidebar at all reasonable?

6.6 Remarks

You may have noticed that Wikipedia articles and technical publications on mathematics are often impenetrable, both symbolically and conceptually. That's reasonable, mathematics is after all a highly technical field of study. We would not benefit here, however, by a purely mathematical approach since one of the primary design principles of boundary math is to help to make math more comprehensible to the non-professional. The strategy is to demonstrate formal techniques within a psychologically motivated and physically friendly system of communication. However, we *are* exploring a formal system, and formal systems inherently describe how a computer works rather than how an organic being thinks.

I've approached this description of James algebra as if we were learning a new language together. Well, a bit more than just a new language, also a new way to think about numbers, a new set of concepts. Comments in the text highlight significant differences between the boundary-based approach and the conventional symbolic approach to notation. A form looks like what it is intended to mean, although it does not necessarily look like how we might interpret it as a conventional number.

Boundary math does not seek novelty as a goal, it seeks simplicity. Most usually, the simplest path is both invisible and obvious. Mathematician Alexander Grothendieck:

> The very notion of a scheme has a childlike simplicity -- so simple, so humble in fact that no one before me had the audacity to take it seriously.[22]

The algebra of boundaries shares with Grothendieck the goal of simplicity, achieving it not by a higher level of abstraction of categories of mappings, but by embracing *only one thing,* the distinction.

Coming up, after introduction to the various James representations and transformations in Chapters 7 and 8, we'll look closely at how counting and arithmetic work in Chapter 9. We'll see the form of addition and multiplication as the interplay between two types of boundaries, `( )` and `[ ]`. And we'll look at the basic structural ideas underneath the natural numbers.

In Chapter 10, we'll take the next big step to introduce the *generalized inverse* and a third type of boundary, < >. Angle boundaries come with a third axiom, so we end up with one axiom that defines each type of boundary, although ( ) and [ ] are co-defined. With James arithmetic in hand, finally in Chapter 11 we will be able look at the entire range of conventional numbers.

We'll generalize the developed tools to include rational expressions and other number types. Then we'll move arithmetic into the higher dimensions of spatial forms. Toward the end of the volume, we'll take a quick look at several other boundary arithmetics.

The [ ] unit in combination with angle-brackets creates a cacophony of non-numeric structures including the exotic and indeterminate expressions of conventional math such as negative infinity, infinitesimal, divide-by-zero, square-root-of-negative-one and logarithm-of-negative-one. Volume III retreats to the uninterpreted James arithmetic to examine the forms that correspond to these exotic expressions, viewing the development of theorems as design choices that come with both strong and weak points. This quasi-mathematical approach is thus a hybrid of rigor and realism. Volume III also focuses on one particular form, [<( )>], with very unusual properties.

Volume II and Volume III each contain a significant surprise. In Volume III it is a *newly resurrected imaginary unit*, one that is more fundamental than $\sqrt{-1}$. In Volume II it is the disclosure that the James angle-bracket is only a *convenient shorthand abbreviation* that allows us to contrast negative and positive and thus remain in familiar cognitive territory where negative numbers are taken to exist. The numeric inverses too are configurations of only round- and square-brackets. In Volume II we'll see that James algebra has only two independent boundaries that are bonded together by one void-based axiom and one rearrangement axiom.

Endnotes

1. **opening quote:** B. Mazur (2003) *Imagining Numbers* p.163. Mazur continues on p.166: "It is also easy to underestimate the difficulties of comprehension that any change of notation presents."

2. **by putting them together into the same container:** Like unit-ensembles, depth-value notation is available to simplify large collections of James units. In Chapter 11 we will see depth-value notation arise naturally as a James form.

3. **out of the containing must likewise be out of the contained:** L. Euler (1802) Letter CIV Different Forms of Syllogisms, (2/21/1761), H. Hunter (trans.) *Letters of Euler* p.406.

4. **Euclid's geometry as the sacrosanct definition of mathematical rigor:** Euclid's *Elements* (circa 300 BCE) is a collection of mathematical proofs that was the second most published book (behind *The Bible*) in Western civilization for over 2000 years. To be a mathematician up until about 1850 was to memorize Euclid's *Elements*.

5. **to be formalized in symbols arranged in sentences and equations:** H. Simon, forward to B. Chandrasekaran, J. Glasgow, and N. Narayanan, (eds.) (1995) *Diagrammatic Reasoning* p.xi.

6. **while the sentential representation does not:** J. Larkin & H. Simon (1987) Why a diagram is (sometimes) worth ten thousand words. In Chandrasekaran *et al*, p.696.

7. **in all schools of philosophy as to the nature of relations:** B. Russell (1923) Vagueness. *Australasian Journal of Philosophy and Psychology* (1) p.84–92.

8. **even the most trivial consequence needs to be inferred explicitly:** J. Barwise & J. Etchemendy (1996) Visual information and valid reasoning. In G. Allwein & J. Barwise (eds.) *Logical Reasoning with Diagrams* p.23.

9. **a potentially confusing distortion of the image of a container:** The overwhelming majority of published analyses of containment address string-based concepts dominated by the right-half (called `open`) and the left-half (called `close`) of a fractured container. Algebraic representation is built upon

a stream of characters, whereas geometric representation occupies space. *Algebraic geometry* has all but banished geometric form from mathematics.

10. **a good notation sets it free to concentrate on more advanced problems:** A. Whitehead (1958) *An Introduction to Mathematics* p.39.

11. **the interpretation of James forms in Figure 6-1:** Multiplication-as-substitution, characteristic of unit-ensembles, will not be used in the following chapters.

12. **many transformations can occur concurrently, all at the same time:** Chapters 4 and 13 include several examples.

13. **the conceptual fantasy of Platonic reality associated with mathematics:** L. Bunt, P. Jones & J. Bedient (1976) *The Historical Roots of Elementary Mathematics* p.122:

> Most mathematicians accept the modern philosophical ideas that their axioms are logically arbitrary and that their theorems are about mental concepts. These mental concepts cannot be actually observed in the physical world. This view of the nature of mathematics can be traced back to the Greek philosopher Plato.

14. **words, insofar as they are adequate, should represent images accurately:** Leibniz to Tschirnhaus, end of 1679 (Math., IV, 481; Brief., I, 405), as quoted in L. Couturant *The Logic of Leibniz,* Ch. 4 footnote 93.

15. **differ depending upon its intended interpretation:** Treating the same symbolic expression in different ways depending upon its *external context* (i.e. its interpretation) can potentially invalidate the use of substitution, equality and identity, making the teaching of mathematics problematic. Symbolic expressions are treated as both clearly defined and ambiguous at the same time. An example of a context dependent system is written language. Both symbolic math and written text are languages. However in the prior sentence what we take as the meaning of the word "language" is determined by its applied mathematical or linguistic context. See E. Gray & D. Tall (1994) Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education* **26**(2) p.115–141. Online 9/16 at `http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1991h-gray-procept-pme.pdf`

In a delightful and often quoted address to mathematics educators, Vladimir Arnold emphasizes that axioms are simply properties of transformations. This is quite similar to the approach taken here, that the pattern axioms identified in Chapter 5 are mechanisms that permit us to identify the absence of difference, i.e. the lack of a distinction.

V. Arnold (1998) On teaching mathematics. A. Goryunov (trans.) *Russian Math. Surveys* **53**(1), p.229–236. Online 10/16 at http://pauli.uni-muenster.de/~munsteg/arnold.html and http://www.math.fsu.edu/~wxm/Arnold.htm Also available at several other academic sites.

16. **share these features with the other boundary systems:** Chapter 14 includes some of these alternative boundary systems. Unit-ensembles, spatial algebra and boundary logic are all described at iconicmath.com.

17. **void-equivalent forms are both blind to sign and blind to multiplicity:** *Void*, of course, has no properties including the absence of both polarity and replication. A void-equivalent form does have a typographic *presence*, however it is a technical error to think that a void-equivalent form is a specific form. Presence is essentially arbitrary and refers to *any* void-equivalent form. Hybrid forms such as ±([ ]) and 2 x ([ ]) tempt us to attribute sign and multiplicity to nothing at all.

18. **sequential concepts that are not relevant to a spatial, parallel form of arithmetic:** Peirce appears to be the first logician/mathematician to suggest that commutativity and associativity are secondary rather than axiomatic concepts.

> (4.374) Operations of commutation, like xy therefore yx, may be dispensed with by not recognizing any order of arrangement as significant. Associative transformations, like (xy)z therefore x(yz), which is a species of commutation, will be dispensed with in the same way; that is, by recognizing an equiparent as what it is, a symbol of an unordered set.

C.S. Peirce (1931-58) *Collected Papers of Charles Sanders Peirce*. Hartshorne, Weiss & Burks (eds.).

19. **only inverse and distribution are fundamental:** Chapter 34 shows that even the concept of inverse is derivative. Network numbers in Chapter 4 and

the James Arrangement axiom in Chapter 8 show that algebraic distribution is a special instance of a broader concept.

20. **Similar to Conway numbers:** J. Conway (1976) *On Numbers and Games.*

21. **achieving a unification of number and logic, of algebra and proof:** This unification has been a fundamental goal of work in boundary math since Spencer Brown.

22. **no one before me had the audacity to take it seriously:** A. Grothendieck, quoted in R. Hersh and V. John-Steiner (2011) *Loving + Hating Mathematics* p.116.

Chapter 15

Next

In the brain there is no principled distinction between hardware and software or, more precisely, between symbols and nonsymbols.[1]
— Francisco Varela (1992)

Our exploration has been conducted on an unlevel playing field. We have been using symbols to raise awareness of the postsymbolic nature of thought, and we have undertaken the exploration within a most unfriendly territory, that of symbolic arithmetic. An implicit expectation has been that the reader is willing to explore iconic form while also considering that the thoughts engendered might be free of symbolic reference. Thought without words, experience without chatter? Distinction, that stuff of minds, requires difference, not reference.

15.1 Choice

How we think about mathematical concepts is influenced by how those concepts are presented and represented. Syntax and semantics, representation and meaning, are tightly connected. In general, how we record and manipulate numbers is a matter of convenience, but the convenience of the *learner* may have been forgotten. For learning mathematics — and for using mathematics — it is more convenient to call upon sensory interaction and natural behavior than it is to manipulate symbols.

A purpose of this volume is to provide evidence that our cultural and academic commitment to symbol processing is a *design choice* and not an inevitability. We have explored two different formal models of arithmetic that are iconic rather than symbolic. Well, we have actually brushed the surface. Under each iconic structure there lies a deep well of potential innovation and opportunity waiting for an intrepid explorer whose appetite may have been whetted by the suggestion that numbers are far more than numerals and symbolic transformation rules. Numbers are contextual relations.

A premiere design concern is comprehension by non-professionals, particularly students. If we did not have to conform to prior instruction, what would be the most desirable way to help students learn how arithmetic works? Just as important as student learning is an overarching question. To what extent have the recent technological and electronics revolutions changed our understanding from a century ago of what arithmetic is? For a society that inundates itself with high density visual information at every waking moment,

It is no longer reasonable to claim that
cognitive skill lies in typographic symbols.

Underlying our mutual exploration is the overt observation that arithmetic is far broader, conceptually and experientially, than what is taught in schools and, indeed, what is thought throughout an academic culture focused on symbolic markings. If indeed math is important to learn, for rigor and for clarity, then surely we must question *which type of math* is important for organic beings. And which dialects are appropriate for the 3D digital age? Does *symbolic rigor* mean preparation for the future, or is it perhaps a history lesson?

Human Nature

A more radical suggestion is that our twentieth century excursion into symbolic mathematics has been a temporary transition at best and a dimensionally degenerate delusion at worst. Each of us is born with evolutionary perspective and with organic knowledge of what might be called **humane math**. Neuroscientist Dehaene is specifically critical of formalist definitions.

> Ironically, any 5-year-old has an intimate understanding of those very numbers that the brightest logicians struggle to define. No need for a formal definition. We know intuitively what integers are. Among the infinite number of models that satisfy Peano's axioms, we can immediately distinguish genuine integers from other meaningless and artificial fantasies. Hence our brain does not rely on axioms.[2]

Mathematics does not necessarily describe nature, it describes our human nature to occlude, to abstract and to simplify. Math is the tool that our culture uses to keep Reality from being overwhelming. Mathematician and historian Morris Kline:

> Mathematics is not something independent of and applied to phenomena taking place in an external world but rather an element in our way of conceiving the phenomena. The natural world is not objectively given to us. It is man's interpretation or construction based on his sensations, and mathematics is a major instrument for organizing the sensations.[3]

There are many thoughts, many sensory modes and many mathematics. The question is not which is right, or even which is better. The question is how do we wish to view ourselves?

15.2 A Hidden Motive

There has been, all along, a hidden agenda, originally initiated by Spencer Brown. The idea is to construct the foundations of logic, numerics and sets, basically all of finite mathematics, from the *same* boundary concepts and forms. Spencer Brown's *Laws of Form* for iconic logic are

crossing
⟨⟨ ⟩⟩ =

Crossing
The value of a crossing made again
is not the value of the crossing.

calling
⟨ ⟩⟨ ⟩ = ⟨ ⟩

Calling
The value of a call made again is the value of the call.

Thomas McFarlane observes fairly that James algebra lacks a clear connection to Spencer Brown's *Laws of Form*.

accumulating
( )( ) ≠ ( )

> The transformative rules and various types of boundaries are introduced *ad hoc* without providing any intuitive basis for them. What is the basis for the adoption of three different boundaries? Are there more fundamental justifications for the axioms governing the transformation of expressions? Is there a deeper connection with Spencer-Brown's arithmetic?[4]

McFarlane brings to attention that there are both mathematical and philosophical motivations in the study of distinction and in *Laws of Form*. In *Distinction and the Foundation of Arithmetic* he derives the James axioms from intuitive first principles. McFarlane's argument is philosophical whereas the presentation herein is structural. It is this structural mapping that awaits a deeper study.

Arithmetic and Logic

Each version of arithmetic in this volume has remained as close as possible to Spencer Brown's formulation of logic, pivoting on a single modification. In *Laws of Form* Spencer Brown constructs elementary logic from two

Arithmetic	*Logic*
accumulation/unification	
( )( ) ≠ ( )	
[ ][ ] = [ ]	⟨ ⟩⟨ ⟩ = ⟨ ⟩
inversion/involution	
([A]) = [(A)] = A	
<<A>> = A	⟨⟨A⟩⟩ = A
arrangement	
(A [B C]) = (A [B]) (A [C])	⟨A ⟨B C⟩⟩ = ⟨A ⟨B⟩⟩ ⟨A ⟨C⟩⟩
dominion	
(A [ ]) = *void*	⟨A ⟨ ⟩⟩ = *void*
reflection	
<A <A>> = *void*	⟨A ⟨A⟩⟩ = *void*
pervasion	
	A ⟨A B⟩ = A ⟨B⟩

Figure 15-1: *Boundary arithmetic and boundary logic*

axioms. The **Law of Calling** generates logical form by suppressing accumulation, while in the arithmetic of numbers Accumulation, the denial of Calling, generates numeric form. Calling allows repetition that does not change value, while Accumulation identifies a specific type of repetition, the repetition of replica units, that in effect *defines value*. The **Law of Crossing** holds for both arithmetic and logic. Crossing defines how value is changed rather than how it is created. Value changes when we cross a boundary.

Figure 15-1 provides a brief comparison. The **logical boundary** is rendered in the figure as ⟨ ⟩. The small differences in the structure of boundary arithmetic and boundary logic provide substantive clues about the foundational structure of finite mathematics. Since boundaries represent cognitive distinctions, the difference in the structure of logic and numeric form specifically identifies the difference between thinking logically and thinking numerically.

Construction

Arithmetic is constructed from logic by these modifications.

Units exist.

( ) ≠   [ ] ≠   ⟨ ⟩ ≠

Arithmetic consists of **two types of units**. One accumulates and one is the same as a logic unit. Logical Truth is a type of numeric infinity.

( )( ) ≠ ( )   [ ][ ] = [ ]   ⟨ ⟩⟨ ⟩ = ⟨ ⟩

Inversion prohibits mutual containment of the two types of numeric unit, as it does for the logic unit.

([ ]) = [( )] =   ⟨⟨ ⟩⟩ =

Nested pairs do not condense, just like the logic unit.

(( )) ≠ ( )   [[ ]] *undefined*   ⟨⟨ ⟩⟩ ≠ ⟨ ⟩

Arrangement is the same for both numeric inversion frames and the logic boundary.

(A [B C]) = (A [B]) (A [C])

⟨A ⟨ B C⟩⟩ = ⟨A ⟨B⟩⟩ ⟨A ⟨C⟩⟩

Dominion defines a hierarchy of existence.

( )[ ] = [ ]

All of this mechanism is designed to maintain accumulation of arithmetic units only, and to assure otherwise that arithmetic units do not get replicated during transformation and rearrangement. Arithmetic units cannot cross a boundary without changing the cardinality of the boundary. Within this constraint arithmetic units are indistinguishable from logic units. When the angle bracket is introduced, it too is equivalent to the logic boundary.

The other substantive difference between arithmetic and logic is that the logic boundary is semipermeable. **Pervasion** asserts that replicas of outside forms can cross a logic boundary to the inside, something that is forbidden for any type of numeric boundary. Semipermeability gives limited permission to replicate without changing cardinality. Not surprising since logic does not support accumulation of cardinality.[5]

pervasion
A ⟨A B⟩ = A ⟨B⟩

15.3 It's Not Easy

Reflection upon the history of the development of mathematical concepts, as John Derbyshire writes in his history of algebra, will "make us realize how deeply unnatural mathematical thinking is."[6] Not only do we have no evolutionary adaptation or propensity for purely abstract thinking, also the greatest intellects of the last two millennia have struggled mightily with what we believe today should be common mathematical knowledge. Here's Morris Kline again:

> In retrospect, this glorification of mathematical reasoning seems incredible. To be sure, tatters of reasoning were employed. But especially in the 18th century when heated debates about the meaning and properties of complex numbers, logarithms of negative and complex numbers, the foundations of the calculus, the summation of series, and other issues we have not described filled the literature, the designation Age of Confusion seems more appropriate.[7]

Symbolic arithmetic is not easy, it has taken centuries for humanity to develop it. John Derbyshire:

> The extreme slowness of progress in putting together a symbolic algebra testifies to the very high level at which this subject dwells. The wonder ... is not that it took us so long to learn how to do this stuff; the wonder is that we can do it at all.[8]

Math Education

The point of demarcation is that students in grade school and in high school and in college do not need to understand arithmetic from the perspective of a foundational mathematician, of which there are only a few hundred world-wide. That would be like insisting that Xtreme athletes import their training regimes into grade school playgrounds, and that children in science classes emulate the life-long research strategies of Nobel laureates. What is misunderstood by mathematics educators is that expertise requires both a vast array of baseline knowledge and extensive training in applying that knowledge. Children, and adults too, are not mathematical experts but the notations, computational strategies and modes of thought incorporated into high school math are intended to emulate those of mathematical experts.

The problem is that professional mathematical tools have leaked down into elementary math education as if they were arithmetic itself. Prior to anchoring our concept of number to sets and to logical theory about a century ago, there was another much simpler arithmetic based on the intuitive Additive Principle. It is this organic understanding of numbers that should be taught in schools. Logic and sets do not have an exclusive right to claim to be the *only* formal foundation for numeric arithmetic.

There is yet another perspective: school math classes do not teach *math*, or at most teach only the tiniest portion of actual mathematics. Mathematician Ian Stewart:

> What mathematics is, and how useful it is, are widely misunderstood. It is not solely about numbers, 'doing sums' as we were taught in school — that's arithmetic. Even when you add in algebra, trigonometry, geometry and various more modern topics such as matrices, what we learn in school is a tiny, limited part of a vast enterprise. To call it one-tenth of one per cent would be generous.[9]

What is taught is not math but rather computer science, the domain that studies and characterizes the behavior of algorithms and automatons. The rigidly structured curriculum materials and the standardized tests of algorithmic skill and technocultural fact is what computer science calls software programming. Math education attempts to download a silicon programming language into organic beings while completely ignoring teaching the skills of symbolic programming.

For an understanding of mathematics, for any hope that future students will be receptive to mathematics, for the possibility of teaching mathematics well, indeed for the possibility of teaching mathematics at all, we need to return to the simple conceptual foundation of mathematics as direct physical experience.

15.4 Summary

Figure 15-2 collects the explicit principles of boundary arithmetic that are scattered throughout the chapters. These principles provide structure for a conceptual map of learning and teaching elementary arithmetic. The principles are stated succinctly so that they may stand alone as bumper-stickers. Some of the phraseology is new, in order to connect the concepts to the central organizing idea of *making a distinction*.

What can be said in summary? Sums are fusions. What is the fusion of the content of this volume? Fusions are wholes. We have drawn a boundary, made a single distinction. Outside there is experience; inside, thought.

You can tell if you understand a mathematics
by changes in your vision.

Inside, where there is nothing, is measured by change on the outside. Outside is identical to the distinctions that we elect to construct.

	VOID	*page*
Void	Void has no properties.	15
Existence	Something is not nothing.	168
	MEANING	
Distinction	Difference is an idea.	15
Calling	Repetition does not make a difference.	372
Crossing	Crossing a boundary makes a difference.	372
Axiomatic	If it is not explicitly allowed, it is forbidden.	140
Communality	When it is shared by all content, it is context.	36
Semantics	A pointer is not what is pointed at.	213
Participation	Meaning depends on how we look.	13
Void-equivalent	Void-equivalent structure cannot make a difference.	151
	ARITHMETIC	
Accumulation	Parts accumulate rather than condense.	171
Hume's	Equality is one-to-one correspondence.	62
Additive	A sum looks like its parts.	3
Multiplicative	Each part of one touches each part of the other.	3
Arrangement	Arrangement is the sole source of complexity.	197

Figure 15-2: *The principles of boundary arithmetic*

From Here

There are two volumes on iconic arithmetic to follow. In Volume II we address just what formalism is, as a mathematical philosophy and as a computational paradigm. We'll go back to visit the original models of arithmetic developed by Frege, Peano, Robinson and others to compare the formal structure of symbolic expressions and iconic form. We have glossed lightly over the concept of **equality**, so we will mold it into void-equivalence. Similarly we have yet to integrate parallelism into our formal model. Then it will be time to address *symbolic and iconic metamathematics*.

We'll explore pure boundary arithmetic, its internal structure and formal consequences without regard to interpretation. Just how can arithmetic fit neatly into one binary relation? What are the structural properties of containment? Just what have we taken on in claiming that arithmetic is about a single physical relationship?

We will be able to dismiss the angle-bracket entirely, reducing James algebra to two boundary types without loosing the integration of operations and their inverses. This of course raises further questions about group theory, since inverses too will be removed from the current theory of arithmetic, reduced to a notational abbreviation. Only Distribution remains, to allow smooth transition between addition and multiplication (and incidentally to permit the generation of mathematical complexity).

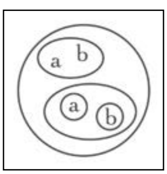
Volume II then feels quite different, with deeper, more challenging questions at the foundations of the current philosophies of mathematics. This is a necessary volume to address the many technical details about the structures, assumptions and thought processes that we now expect grade school teachers and students to grasp intuitively.

In Volume III we'll tackle the neglected topic of the empty square-bracket and its possible interpretation as an infinity. What are the consequences of mixing numeric and non-numeric units? How does James algebra handle division-by-zero and indeterminate forms and exotic bases? These noxious concepts are in the interpretation, but what are they in the form? We will be able to organize the indeterminate forms such as $\infty - \infty$ and $\infty/0$ into a single coherent pattern. The Mother of all imaginaries, -1, is implicated with every strangeness that occurs in arithmetic.

Volume III returns to pure exploration of form by examining `J`, the logarithm of `–1`, as the foundation of imaginary numbers from which our current compound imaginary `i` can be derived. Just what does the imaginary numeric realm mean, and what are its fundamental structures? We'll take a close historical look at Euler and Leibniz and Bernoulli as they invent complex numbers yet fail to converge on the meaning of `J`. We will see the features of the imaginary domain arise out of a simpler foundation guided by patterns of containment.

We'll also develop the James of calculus derivatives, revisit the oscillation of logarithmic and exponential levels of nesting in the context of imaginary forms, and take a closer look at the concept of mathematical morphism. And then we'll unify π, trigonometry, cyclic behavior, hyperbolic functions and complex logarithms naturally within iconic form.

Phew! And hopefully a solid foundation for beginning to understand, in potential Volumes IV and V, Spencer Brown's reconceptualization of *logical thought*. Boundary logic itself provides a far more revolutionary reconstruction of the nature of rationality than does our current exploration of boundary arithmetic. Logic is unary not dualistic; `FALSE` is a void-equivalent concept that can be completely disregarded and discarded. Deduction means to identify and delete void-equivalent forms. The path to critical thinking (as well as to new computational architectures) is through removal rather than accumulation of structure. The basis of rationality is emptiness.



15.5 Remarks

Spencer Brown's book *Laws of Form* is seminal, but in the fifty years since it was written our experience with boundary logic has grown significantly. There is an extensive collection of papers describing boundary logic at `www.iconicmath/logic/boundary/`.

This volume is if anything pragmatic. And yet some philosophy shows through in the form of the metaphysics of *void*. There are two voids. One we can talk about; that one is indicated by its boundary. That void is coupled to distinction as the foundation of unity. The other void is destroyed upon mention. The unmentionable is the metaphysical motivation of this volume. The concept of number is already so far removed from its origin that it is essential to regress backwards, from Two to One to Nothing to Silence, in order to find number, unity and absence.

Endnotes

1. **opening quote:** F. Varela (1992) *Ethical Know-How* p.54.

2. **the brain does not rely in axioms:** S. Dehaene (2011) *The Number Sense: How the mind creates mathematics* p.223.

3. **mathematics is a major instrument for organizing the sensations:** M. Kline (1980) *Mathematics The Loss of Certainty* p.341.

4. **Is there a deeper connection with Spencer-Brown's arithmetic?:** T. McFarlane (2007) *Distinction and the Foundations of Arithmetic.* Online 6/16 at `http://www.integralscience.org/tom/`

5. **logic does not support accumulation of cardinality:** For an extremely concise summary of the growth of arithmetic and logic from nothing at all, see W. Bricken (2006) The Mathematics of Boundaries: A beginning. In D. Barker-Plummer *et al* (eds.) *Diagrams 2006,* LNAI 4045 p.70-72.

6. **how deeply unnatural mathematical thinking is:** J. Derbyshire (2006) *Unknown Quantity* p.40.

7. **the designation Age of Confusion seems more appropriate:** Kline, p.169.

8. **the wonder is that we can do it at all:** Derbyshire, p.51.

9. **to call it one-tenth of one per cent would be generous:** I. Stewart (2011) *Mathematics of Life* p.8.

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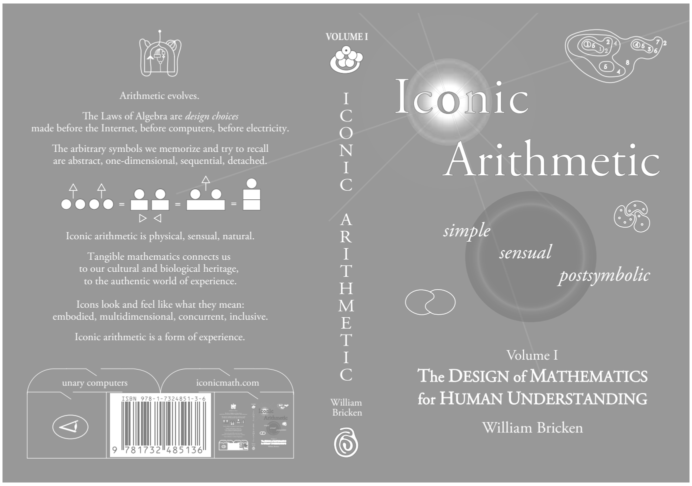

Cover Words

Arithmetic evolves.

The Laws of Algebra are *design choices*
made before the Internet, before computers, before electricity.

The arbitrary symbols we memorize and try to recall
are abstract, one-dimensional, sequential, detached.

Iconic arithmetic is physical, sensual, natural.

Tangible mathematics connects us
to our cultural and biological heritage,
to the authentic world of experience.

Icons look and feel like what they mean:
embodied, multidimensional, concurrent, inclusive.

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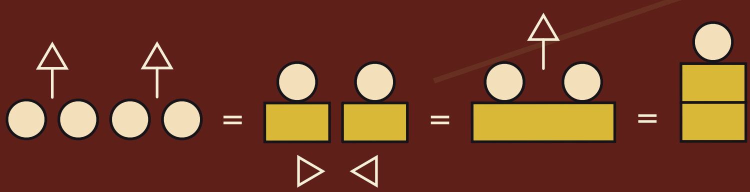

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