CHAPTER 2 AUTHENTIC MATH

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Authentic math is the kind of math that we come across naturally every day. We find it in common contexts. Within those contexts it makes intuitive sense. Authentic math is not obscured by cryptic symbols, it is not tangled into intricate or tricky puzzles, it does not require advanced training. In fact, most people find authentic math easy. Most people can keep track of the money that they spend each week. Most people can cut a birthday cake evenly into eight pieces. Most people know when it is time to buy more milk. In contrast, when these same tasks are presented abstractly on a written test in a classroom, most people abandon their native wisdom, freeze up, and fail. Authentic math and school math are distinctly different skills.

Math is multifaceted, it is a discipline and a skill, a tool and an activity and a social value. I'm a math teacher, so I think about math education. My focus is not on the kind of math that professional mathematicians do, whether they work as technicians analyzing data or as theorists exploring pure abstraction. Nor is it on the kind of math that math teachers usually teach. In fact I think that teaching and learning algorithms like long division and how to add fractions is equivalent to teaching and learning arcane magic rituals that work only for True Believers. Math is a magic amulet that only those who "get it" get. The "it" is an understanding that school math is a game. Some humans like doing those meaningless manipulations. There are contest rewards and scholarships for those who do it well. A century ago, it was a human profession.

I teach math as a conceptual skill rather than a computational chore. I teach how to use math as a tool in non-technical situations. Not to "do" math, but to use a calculator or computer to do the math that may be relevant to n authentic situation. We can learn a lot just from looking at what people do when they do math outside of school. Non-professional math is a daily activity, it occurs in brief flurries: scoring a recreational game, paying at the check-out, weighing ourselves, fitting a large box into the car, getting to the movie on time, restocking the refrigerator, catching the flaw in an argument, gathering statistics on a football player, solving simple problems such as figuring out gas mileage and checking the bank balance. It is puzzling how much our society values math, so much so that every one of us must study it in school for a decade. But here's what is extremely weird: we assiduously avoid teaching any non-professional math. We teach abstract professional math, the kind that takes years of preparation to understand precisely because it is pathologically disconnected from our bodies. It is bewildering that some people believe that the world is organized by math, that math preceded humanity. It's plain scary to contemplate that math teachers are willing to subject us to many years of indoctrination and humiliation so that we may be able to add meaningless fractions together. 7/33 + 8/21? Come on! There is no authentic situation where this may occur. Try to find one.

Math has been constructed to embrace specific values. Unlike literature, the language of math does not permit us to write about ourselves, to engage in metaphor or allusion, to be nuanced or ambiguous, to be cooperative or obstinate, to consider context or meaning, to convey emotion or instinct, to incorporate gestures or physical movement, to embrace blunders or errors, to balance resources or commitments, in short, to engage in being human. It may be a good idea to examine why we consider math to be so important, especially since it is so far removed from human values. It has certainly been a challenge for me to try to find humane values within school math, the type of math taught in 90% of the math classes in the US.

What would it mean for math to be humane? It's pretty simple. *Humane math is the math that people use everyday.* It's math that is easy to understand and easy to use. Minimally, studying math should not harm those who study it. Humane math education would build upon the intuitive and common sense skills that we call upon naturally, and would be expressed in a language that we understand naturally. Humane math is authentic math expressed clearly. With a few changes, all of the math that we learn in school can be made humane.

AS UNREAL AS IT GETS

Try this: Close your eyes and estimate how much it would cost to replace the food in your refrigerator. Consider each item on each shelf and add up the total amount that you would have to spend (Figure 2.1).

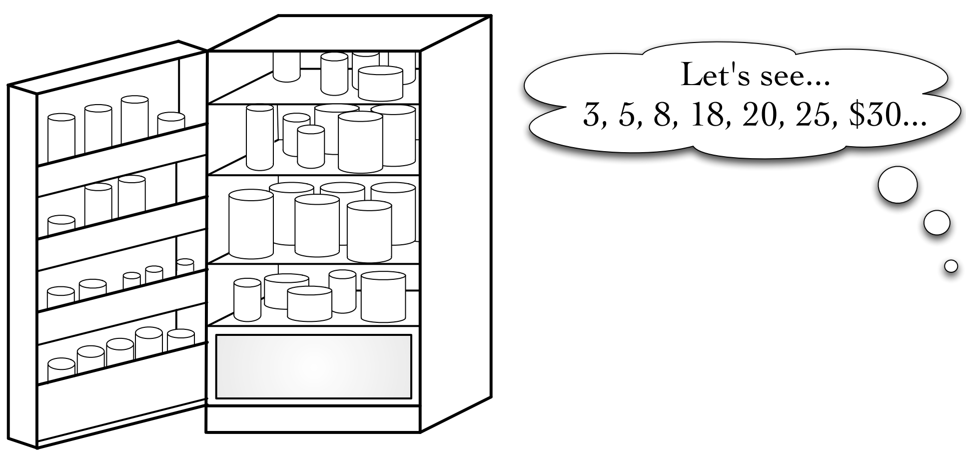


Figure 2.1: How much would it cost to replace the food in your refrigerator?

Now try this: Add up these numbers,

```
  1.39
  2.98
  2.08
   .98
  1.35
  1.95
  8.00
  3.15
  4.56
  1.67
  1.66
+ 1.66
```

Permit me a guess. Chances are you'd be willing to try the refrigerator task, and chances are that you would resist the column addition task. There is nothing remotely appealing about column addition, it is rote, mechanical and tedious. The refrigerator task is more work, but it is grounded in reality and it may spike your curiosity. The effort might be worthwhile because the result would have personal meaning.

The way we go about the two addition tasks is fundamentally different. The refrigerator task calls for quick estimation. The other task calls for the rules of school column addition. Good adders on school problems do not follow the school rules as they are taught. The algorithm of addition is taught linearly as if the numbers were being fed to a machine. This indeed was the origin of column addition, before computers, humans were paid to be computers. There are a multitude of short cuts and tricks that make column addition easier, they are often considered to be cheating. Most people will interpret the column of numbers to be dollars and cents, for example, grounding the raw abstract decimals with some meaning. We may see 2.98 as "three dollars less two cents". We may recognize the last three numbers in the column as "three for five dollars". In contrast, while adding up the refrigerator goods very few of us would use $2.98 as a cost, we would think "that's about three dollars". We would add nice simple whole numbers to reach a pretty good approximation.

Out of curiosity, I did the refrigerator task. Interesting. Hundreds of dollars. But I didn't do the exact column addition task, its visual appearance alone achieves the objective. The column of numbers looks as meaningless as it is. On a five second scan I came up with $31. (Oh, but my answer is actually 31.00, because there is nothing about dollars and cents in the problem.) My quick scan is within 2% of the exact sum. If these were the prices of the items in my refrigerator, I wouldn't care about the 44 cents, the raw .44 in the exact sum of 31.44. The quick scan is good enough for almost all purposes, certainly good enough to answer my curiosity about how much I have invested in cold food. I'd venture to say that the only time that the quick scan is not good enough is when you are sitting in a math class.

Exact and Authentic Numbers

School math teaches that numbers are exact. Applied math recognizes that numbers are approximate, and that actual values depend greatly upon the context. $2.98 in school math is three dollars in our minds, more than three dollars at the check-out when tax is included, and less than three dollars if we have the right coupon. The $2.98 shelf price is approximate, even when it looks to be exact. When raw numbers are associated with meaningful events, the numbers themselves take on different meanings. The problem is that school math does not teach estimation, nor does it teach how context influences value. School math does not teach how to use math authentically.

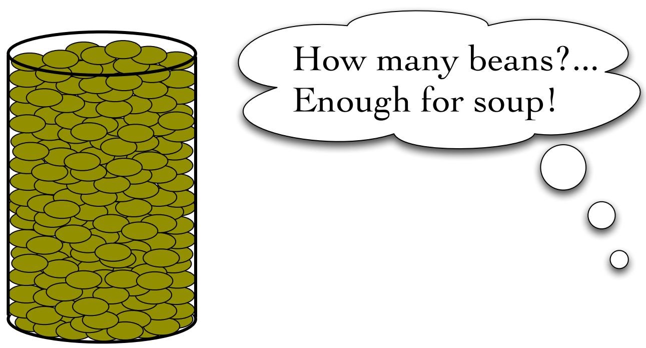


Figure 2.2: How many beans are in the jar?

How many beans are there in a jar (Figure 2.2)? School math teaches us how to count the number of beans exactly. But how we count them depends on why we count them. It is almost inconceivable that we would need to count them at all, outside of some counting game. What matters is not the number of beans but whether or not there are enough beans for whatever we plan to do with them. Counting the beans will usually end up with one of two answers: enough or not enough. And for that, we don't need an exact count.

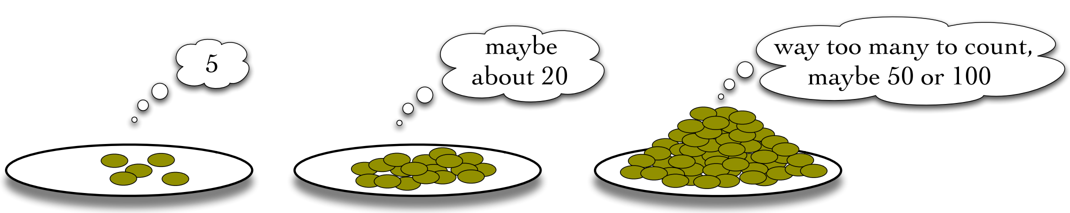


Figure 2.3: How high should we count?

When faced with column addition, a context-sensitive approach would have us ask: "How is the answer going to be used?" "Can we get within 10 or 20 percent within a few seconds?" Then we would ask: "Is that good enough?" Of course, good enough depends upon the context of use. The main idea is that an authentic context rarely requires exact counting.

People count *things*, and we count only small numbers of things (Figure 2.3). The exactness of any number beyond about 100 is not important, except in very special circumstances like taking 100 pennies to the bank to exchange them for a crisp dollar bill. Even then, it is only the bank that cares, I certainly wouldn't lose any sleep if I handed over 101 pennies to the clerk. "Keep the change!" In Washington State there is a business that places huge coin counting machines in grocery stores. People bring in their piles of change and pour them into the bin. The machine whirrs for a while and tells you how much you gave it. It then gives you back 90% either in bills or as a credit at the market. Apparently most folks don't mind if they hand over 110 pennies in exchange for the convenience of a dollar bill.

What Math Looks Like

Humane math is math that is anchored to the physical world, but it is broader than what is called Applied Mathematics. In addition to being grounded in reality, humane math also requires that math be customized to support human understanding. It must be expressed in a form that is easy for people to use and to understand. Human-friendly math involves the body and the senses, it is visceral as well as cerebral (Figure 2.4).

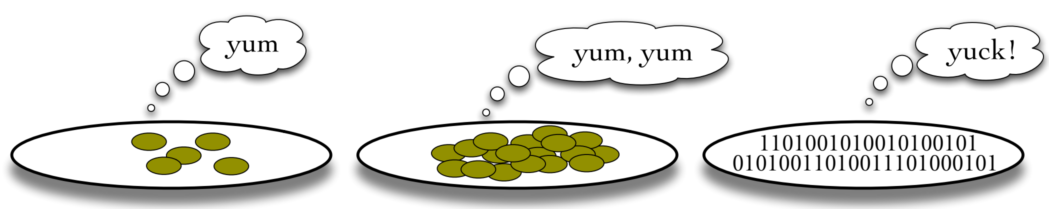


Figure 2.4: Should math look like the world, or should it look like symbols?

Prior to the twentieth century, arithmetic was done using fingers, tallies, pebbles and beads. The Latin word for pebble is *calculus*. Math was a tactile and visceral physical act that was supported by an occasional symbolic form. Math notation is an accumulation of jots and ticks that has constantly changed over centuries. None of it was designed for ease of use. At the beginning of the twentieth century, mathematics was placed firmly onto a symbolic foundation that consists of strings of digits and signs. It was designed to be used by people with extensive training. The formalization of math makes proofs easier to follow when they are written and checked by hand, and it makes automation of computation easier for machines.

The hallmark of *symbolic math* is that the association between symbols and their meaning is arbitrary. The meaning of a symbol must be memorized. This works exceedingly well for computers, because computers do not incorporate meaning into their computations. Computers follow the rules of symbol manipulation, rules that address only the sequence of the symbols, rules that are independent of the meaning that the symbols carry.

The math that people use does not need to look like the math that computers use. Modern computers operate on binary streams at incredible speeds. Computers use different mathematical techniques and algorithms than those taught in school. We should not expect people to find the manipulation of strings of digits appealing, but the problem is that symbolic strings, the kind used in writing and in numerical computation, have co-evolved with our culture. We believe that all people should be facile with these strings. That's what the three Rs of education are all about, learning the symbolic codes of reading, writing, and arithmetic. Math does not need to look like what it looked like before computers, when people had to do mathematical computations by hand. The original "computers" were people, folks who computed for a living. That job no longer exists. Today, computation by hand is obsolete but school math has not changed. In this century, computer software permits us to customize how math looks, and particularly how it can best be presented to support human understanding. Today complex calculations can be shown as dynamic simulations, such as weather maps, GPS displays, diagrammatic patterns of animal migration, and flow charts of the distribution of goods. These simulations are interactive spatial displays that engage our bodies and our senses, they are what math looks like when it is intended to be understood (Figure 2.5).

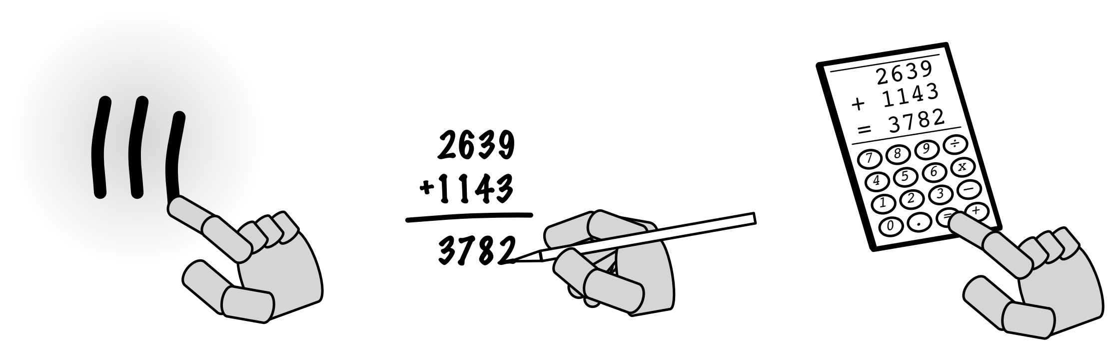
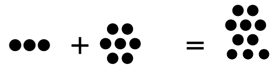


Figure 2.5: Find the historical artifacts.

Humane math is *iconic* rather than symbolic. Mathematical icons are intuitive, obvious representations of mathematical concepts. Icons look like what they mean, they guide us to obvious behavior. The meaning of a symbol is assigned without regard to the appearance of the symbol, but the meaning of an icon is apparent in what it looks like. •••, for example, is an iconic representation of three because there are three dots. There is nothing to memorize, the meaning of three is encoded directly into the shape of the iconic three. The numeral 3 is both symbolic and iconic (count the prongs), a residual from when all numerals were iconic. The numeral *7* is purely symbolic, it has no features that resemble its intended meaning. Not so with  . Iconic numerals show us how to add them, there is nothing to learn. To add iconic numbers, put them in the same place.



It is possible to express all commonly used math as icons, making math visual and making math operations tactile, manipulative, and easy to understand. Iconic math subverts symbolic abstraction by returning it back to earth. It is quite easy to translate a human-friendly math notation into computer-friendly strings of zeros and ones, without loss of rigor or accuracy. It is terribly difficult to make strings of digits palatable to humans. The big step, what is deeply needed, is to wean schools from their dependence on symbolic form.

Actual and Virtual

There are deeper concerns than what a number looks like, but we must first deal with a very delicate issue: the difference between actual and virtual. The issue is delicate because we humans are exceptionally facile at switching our attention between information coming from the physical world and information coming from our internal processes. The need for a distinction between actual and virtual has been compounded by the digital revolution. We are immersed in a plethora of electronic devices that shower us with virtual, digitally-mediated information. All information from TV and computers and radio and film and books and video games is virtual. The images and sounds they provide are not generated by the natural world. To call upon an analogy from the feeding behavior of birds, media provides predigested experience. It masticates reality and regurgitates an imaginary cud. Similarly, math itself mediates between the real and the imaginary. Digital-mediation is mathematical-mediation, they are the same thing. When digital/mathematical mediation disconnects us from direct interaction with reality, our experiences are *virtual*. Mediated activities occur in *virtuality* rather than in reality. In contrast, when information reaches our sensory systems directly, without digital/mathematical mediation, our experience is actual, physical, *authentic*. Compare listening to a recording of a band to listening to a live performance. "Live" does not mean that the band is live when it plays, it means that we participate by bringing our bodies to the performance. A recording is digitally-mediated, it permits us to remove our bodies from the real experience. No digitally-mediated experience is authentic, just as no mathematical symbolism is authentic.

To digitize means to convert an authentic situation into a mathematical pattern. In digital-mediation the real world is first encoded into a stream of zeros and ones. The stream of digits is not real, although the on/off switching of current in wires inside a computer chip and the oscillations of the electromagnetic field that reify the digital stream are real. Electronic devices then decode these binary streams into sounds and images that our physical senses can respond to. But there is a very big gotcha: the digital stream can be arbitrarily modified. *What comes out has no requirement to resemble what went in*. Digital-mediation converts real into imaginary, with no covenant and no capacity to return to reality [1]. When we digitize, authenticity, meaning, context, what we casually consider to be reality, are all lost. The digital form is well suited for transmission from the broadcast studios to our homes, but digital transmission also disassociates authentic experience from its context and delivers at best only a simulation of reality, a reality show, a virtuality. The delicate point is that we have become so accustomed to digitally mediated experience that we no longer acknowledge the difference between authentic and mediated activities.

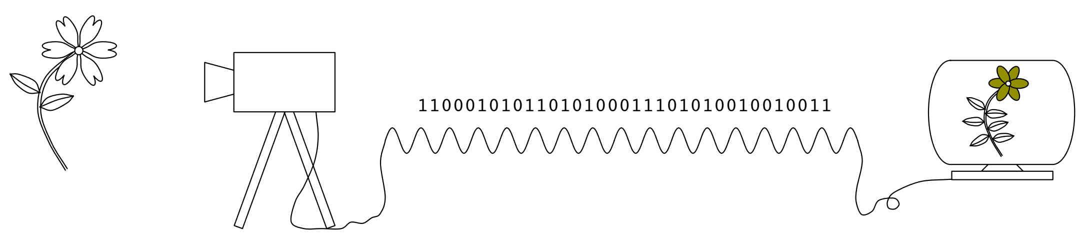


Figure 2.6: Mathematical/digital mediation turns real into imaginary.

Mathematical mediation disconnects from reality (Figure 2.6). Only professionals care about the streams of zeros and ones used in digital transmission. Similarly, only professionals should care about the technical details of mathematical computation prior to its conversion into useful information. Currently our math classes teach students about the stream of ones and zeros, while ignoring both what is being broadcast and what is being received.

Math Is Not Real

What does it mean that math is not real? It certainly does not mean that we cannot use it, or that it is not helpful. Math is not real simply because it is virtual, it cannot be physically experienced. This perspective contradicts the dominant societal illusion that math is somehow out-there in the Universe, that math is somehow real. It is not. Math is something that we humans superimpose upon authentic situations so that we can benefit from the conveniences of abstract description. Math does not describe reality. It does let us describe a particular simplification of reality, one in which meaning and context and physicality are ignored. When a stick-chasing dog gauges its trajectory to intersect with the arcing parabola of a thrown stick, the dog does not do any math (Figure 2.7). Neither does the stick. Math itself is a digital-mediation, a virtual description of the paths of stick and dog. But all of the story, all of the context and circumstance, must be omitted. Math describes a perfect, imaginary world, a world without detail, without history. In using math, we elect to ignore the actual world in order to focus on our fantasies of simplicity and perfection.

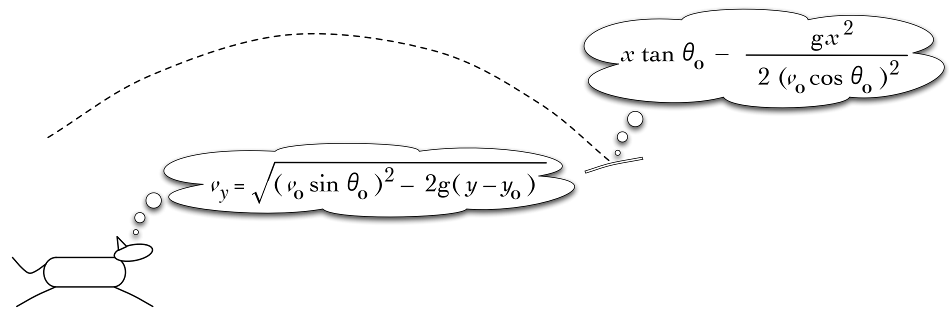


Figure 2.7: Math is not in Nature.

When a planet follows Kepler's Laws as it orbits the Sun, it is not computing where to go next. Kepler's Laws are virtual, they are convenient digital descriptions that we humans use to better understand orbital dynamics. The planets do not follow Kepler's Laws when we look closely enough. Kepler's Laws worked well in the seventeenth century, but in this age, we have found that orbits are far too unstable to follow human laws. At least that's what up-close observation is telling us. Kepler and Newton believed in a perfect universe, one that was crafted by a divine agent and that works flawlessly, a clockwork universe within which every turning gear and every tick unfolds within an ultimate choreography of infinite precision. This was a universe within which mathematics not only could rule but must rule. Today, we know that we cannot predict the orbits of three gravitational bodies, much less thousands. It is not that our math is too weak to describe heavenly motions, it is that current mathematics predicts that planetary behavior is unpredictable. How well a mathematical description works depends on how closely we choose to look and how much we are willing to ignore. All too frequently, we call a mathematical simplification "law", and then proceed to confuse the description with the thing itself. This is the classical error identified by general semantics: "The map is not the territory."

Context and Meaning

Human knowledge is *situated*, knowledge gains meaning from how it is used, and it loses meaning when isolated from human activity. We bring to every use of math a context, a belief about what we are doing and why we are doing it. Context helps to define what a symbol means. Computers however manipulate symbols regardless of context and regardless of meaning. This is actually quite clever, we get extensive powers of computation without having to worry about making sense. When we enter numerals or other symbols into a calculator, we step away from the circumstances that suggest the computation. We reconnect to circumstance when the calculator displays the result of its computation. Calculators don't care what we type in, what matters is whether or not the calculator does what we expect it to do. Similarly, we do not care about how a calculator goes about getting its result, what matters is the result itself. Silicon calculation occurs at an unimaginable scale, along wires that so mall that comparing them to a human hair is like comparing a person to the Empire State Building.

It is quite important that calculators and computers do not dabble in meaning, because the meaning that people assign to mathematical symbols is more subtle, more nuanced, more context-sensitive than the math alone can express. Computers are unable to keep track of human circumstance. In a subtle ploy, the internet is simply redefining human circumstance as what we do on the internet. The flip side is also true, humans cannot effectively do the computational work of calculators. Without meaning and context, computation is intentionally not for human consumption.

AUTHENTIC MATH EXPERIENCES

My agenda is to describe the look-and-feel of humane mathematics. This agenda is broader than finding new and better ways to teach or to learn math, it is also to ground math in forms that people can understand. Math is virtual but this does not need to be the case. In fact, common math before the twentieth century was authentic, the virtualization of popular math is recent. In the early twentieth century, the discipline of mathematics rested on a very insecure foundation, full of contradictions and doubt. The solution was to build the foundations of math without any reference to authentic situations, without any connection to reality. This agenda includes denying that people invented math, and denying that math needs to be connected to our bodies in order for it to make sense. Now that we are experiencing the consequences of this decision, now that most people literally hate math, it is time to return math to reality, to the body of our experiences and to our bodies which are the source of experience.

Math is truly a wonderful invention. It provides a useful way of thinking about some things for almost all people, and it provides deep and thrilling fascination for a few people who love its beauty. Math as an art is right up there with painting and recorded music. It is nowhere nearly as important as understanding and speaking a language.

I assure you that I am not questioning the concept of math or its desirability or its beauty. I'm questioning its implementation as a social skill. I am saying only what teachers and students already know, that math is not real. Math is the foundation of much of our culture's physics and technology and commerce; it has contributed at the most fundamental levels to the advancement of our civilization. It is useful, yes, as are many of the fantasies that we support. Every imaginary creation, whether it be the invisible play friend of a four-year-old, or the little white lie we tell to save someone's feelings, or the thrilling saga of a distant land in a galaxy far far away, every imaginary fantasy provides some actual value. The fantasy of mathematics also provides great value to our society. That does not, however, make it any more real than the thoughts going through your head right now, or the meaning of the words that are typed on this page.

Most people believe that math *is* real. This is perhaps driven by a deep misunderstanding and miseducation about what math actually is and how math actually works. Mathematical thinking suffers from *symbolic detachment*, from our collective delusion that math is not required to connect to the same reality we live in and from our collective belief that math constitutes a valid reality. The symbolic structure of mathematics was decided upon well before ecological awareness and holistic thinking became necessary. Current mathematics does not and cannot address the symbiotic unity of the earth; math is, after all, idealistically detached from the world we live in. Symbolic detachment requires that we ignore intricate ecological codependencies because mathematical symbols attach only to isolated, discrete, and stable objects and processes. Math does not possess the conceptual structure to help us to prepare for the interrelated, context-dependent, ecologically-unified physical challenges we are facing in this century. Certainly new math will be born -- that is one of the themes of this book -- new ways to look at math will flourish and perhaps help us to understand the holism of diversity so profound that every part is unique.

We'll next look briefly at authentic counting, addition, geometry and logic, and elaborate upon them in later chapters. Each type of math can be expressed elegantly using icons, each can directly illustrate what it intends to mean. I'm indebted to the brilliant work of Brian Rotman, whose series of books deconstructing the nature of mathematics taught me about authentic arithmetic [2,3,4]. His *Ad Infinitum* is subtitled *Taking God Out of Mathematics and Putting the Body Back In*. I learned about authentic logic from George Spencer-Brown's seminal work *Laws of Form* [5], and about authentic geometry from Benoit Mandelbrot's seminal work *Fractals: Form, Chance, and Dimension* [6].

Authentic Counting

Consider the simple act of counting using the whole numbers.

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1, 2, 3, ...
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What does the ellipsis, "...", mean? We say that the natural numbers go on forever. But forever is a metaphysical concept, a doctrine of belief. It is precisely a theological claim, with no basis in actual experience. There is no possibility that any authentic thing might "go on forever". There is no one who can do this type of counting, no one who can verify that it is possible to count forever. Even if every person who ever lived counted for every second of their lives, we could not reach forever. If every computer on earth were put to this task, running at blinding gigahertz speeds for as long as our civilization might endure, we would not succeed. No matter how much we progress in counting upwards, every number is equally far away from infinity (Figure 2.8). Infinity is certainly a valid virtual concept, but it can never be a humane concept because it is divorced from human experience. The ellipsis, ..., stands in place of *utterly impossible*.

We do not say, "There are enough natural numbers for any purpose." Instead, we teach that there are an "infinity" of these numbers. We do not teach that 10^{100} (i.e. a google, 1 followed by one hundred zeros) is not a number, although it is without any meaning. We say instead that it is a number! That is, numbers are not held to any form of accountability, they do not need to be grounded, they do not need to have a meaning, they do not require resources or maintenance, they just are. This perspective is called abstract or theoretical, I've been calling it imaginary or virtual. We are teaching theoretical mathematics in the first and second grades, where it is called "back-to-basics".

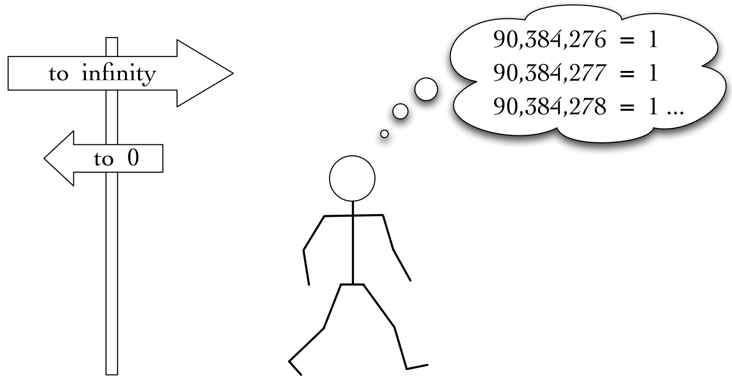

Figure 2.8: On the road to infinity, we are always at the beginning.

Our belief in simple counting is based on the illusion that it is possible for something (the natural numbers in this case) to exist without need of resources, without taking up time and space, and without authenticity. When we require that integers apply to what is real, the theory of counting no longer works. For example, one million dollars plus one dollar will still equal one million dollars in the eyes of everyone except professional money trackers. Even for something as important as counting votes in a public election, the accuracy of a count is about 1 in 10,000 [7]. And even when the physical ballots are all in the same room, it is extremely difficult to count them by-hand accurately. It is just as difficult to find an occasion where exact counting past, say, 100 is a necessity.

Exact Numbers are Inaccurate

Exact numbers, school numbers, will always reach a point where they are inaccurate. Not inaccurate in the school sense of not-being-exact, but inaccurate in a much more significant sense: they do not apply to reality, they give us an inaccurate picture of the world we live in. When we place a ruler against an object to measure its length, we align one edge of the object with the end of the ruler. This alignment is of course never exact, it is usually good enough. We measure the other end of the object. We can look closely to get a fine grain measurement, and we can use sensitive instruments to generate still more measurement accuracy. But we cannot measure exactly (Figure 2.9). There comes a point at which the marks on the ruler are insufficient, where we have to approximate. Ultimately this constraint may come down to the roughness of the ends of the object, or an inability to hold the ruler exactly still, or the thickness of the lines on the ruler obscuring a closer measurement, or thermal fluctuations causing the ruler to expand and contract. Inaccurate measurement is not by choice, it is unavoidable, it is part of Nature. This is an essential difference between authentic and theoretical. Math is approximate whenever it is applied to reality; math is exact only so long as it is not associated with what is real. Ironically, the pure math of quantum mechanics does not permit exact knowledge of anything actual.

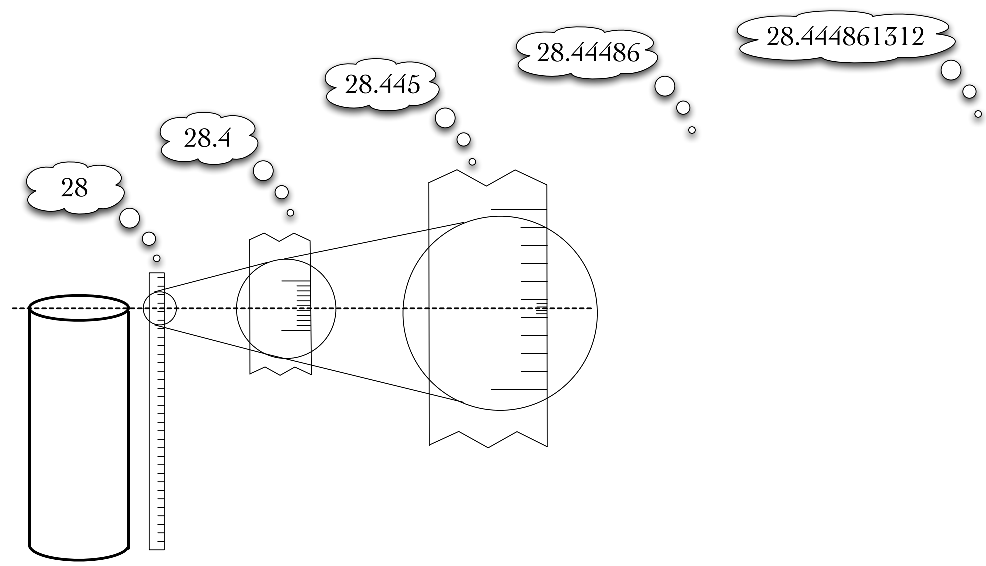

Figure 2.9: How accurate is a measurement?

Once we acknowledge that all measurements are inexact, that all authentic numbers have an associated margin of error, we can immediately see that counting cannot work forever because at some point we reach numbers so large that we can no longer assure their accuracy. A very large specific integer, say something like 1,204,654,829 (one billion plus change), cannot have an authentic meaning because it is impossible for humans to count up to a billion without making mistakes. When numbers are huge, it is impossible to know that we have exactly that many. There is a kinder perspective: we do count accurately, we do not make mistakes. Rather, *accuracy* does not mean perfect exactness. While accuracy is a phenomenon of our actual world, exactness is a construct of the virtual world. Exactness, like perfection, is a theological concept, it is not actually possible. Exactness is a belief rather than an observation.

People are unable to count a large number of things exactly. As well, it is extremely unlikely that a large number of things will remain stable long enough to be counted. In the time it takes to say the exact number of people on Earth, that number has changed. We eat about 35 billion chickens each year, but it is impossible to state at any one time exactly how many chickens we have eaten. How specific a large number can be depends upon what is being counted, how fast it is changing, and how accurately we are able to count or to measure. How specific a count *should* be depends upon how the count is used, upon why we are counting. Exactness becomes increasingly unlikely the larger a number becomes. The limit of exactness is about ten thousand in all but very special circumstances. The limit of exactness that matters is usually much less. The most accurate scientific measurement we have includes about fifteen digits of accuracy. No physical measurement can be accurate past say 20 or 30 digits. Physics itself puts quantum mechanical limits on the possible accuracy of measurements. No number can reflect reality with arbitrary precision because the structure of physical reality itself does not permit it. The problem is not inaccurate counting, the problem is our belief in a perfection that does not and cannot exist.

Another fundamental characteristic of the abstract whole numbers is that the distance between two successive numbers remains the same, regardless of the magnitude of the numbers. The space between 1 and 2 is the same as the space between 5 and 6, between 39 and 40, between 13413 and 13414, ... forever. The rungs of the ladder of counting are all equally spaced because that is a quite convenient way to construct abstract numbers. Now, if I have $1 and a sandwich costs $2, then the space between my wealth and my ability to eat is huge. Should I have $13413, the extra dollar to arrive at $13414 will never be as important. To follow the logic of "...", the difference between 341,628 bananas and 341,629 bananas is the same as between 0 bananas and 1 banana. This makes no sense in any authentic circumstance. The authentic distance between numbers becomes smaller as the numbers get larger (Figure 2.10).

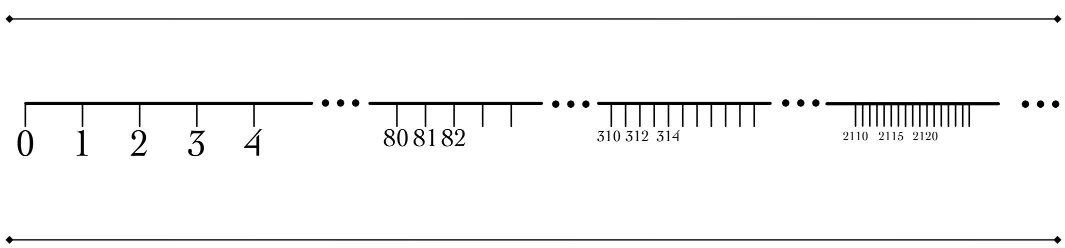

Figure 2.10: The authentic distance between numbers varies.

Why Count?

We need to stop to ask: just what is the purpose of counting anyway? Is it to conform to some fantasy ideal, or is it to help us to understand our world and our place within it? Counting is a measurement when things are being counted, it is a fantasy when numbers are reeled off in succession without referring to something.

Authentic counting follows different rules than the imaginary counting taught in math class, because authentic counting includes counting error. Said the other way around, exact counting does not work in authentic situations. To confuse imaginary counting with authentic counting is the epitome of incorrectness. It is saying, “Never mind the cars, the road is a great open space to play.” Imaginary counting teaches young children how to be wrong about the world they live in. It teaches that disconnection from authentic experience is desirable, admirable and necessary (or you will fail math). It encourages belief without possibility of verification. Abstract numbers come from a place not of this world, a place where space and time and effort are all non-existent. A world in which you can say 14,634,578,222,435 and mean it. But that is not the world we live in. More importantly, the infinite accuracy of abstract counting teaches us poor habits of thinking, of visualizing, of expecting, of measuring, of being responsible citizens in a technological society.

Tally arithmetic, what later in this book is called *unit ensemble arithmetic*, began at the dawn of culture. *To tally* is to make a mark that corresponds to an object. It is a way of using the principles of counting without having to count. Tallies are fundamentally grounded, each physical mark matches an actual object. Tallies require one-to-one correspondence between existent marks and existent things, but which mark matches which object does not matter. Any pairing will do. There are no infinite tallies, there is no concept of distance between each tally mark. We have abandoned this skill only recently, in favor of symbolic strings that are more trusted for mathematical proof. What has been overlooked is a formal theory of tally arithmetic, a way to show that common sense physical math is also rigorous, a way that math can be both abstract and concrete at the same time.

Authentic Addition

To add is to put things together, usually in a common space. Adding together means the same for all things, but the mathematical interpretation can vary profoundly.

One lump of clay plus one lump of clay is still one larger lump of clay. Putting lumps together, adding them, is not the same as counting how many lumps (Figure 2.11a).

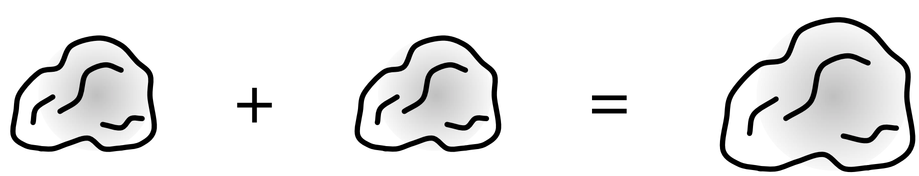

Figure 2.11a: For lumps of clay, 1 + 1 = 1

One cup of water plus one cup of sugar does not yield two cups of sugar-water. Sugar and water can share space. Adding sugar and water together does not conserve volume (Figure 2.11b).

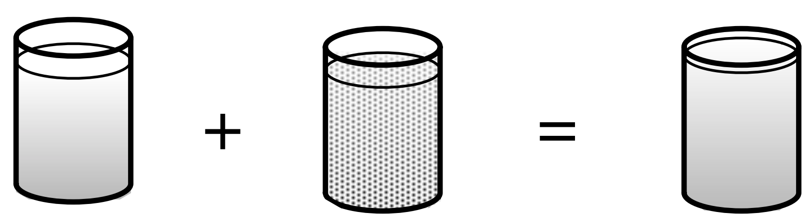

Figure 2.11b: For sugar and water, 1 + 1 = 1.1

When two cups of water are added together in the same cup, nothing changes. You can't add more to something that is full (Figure 2.11c).

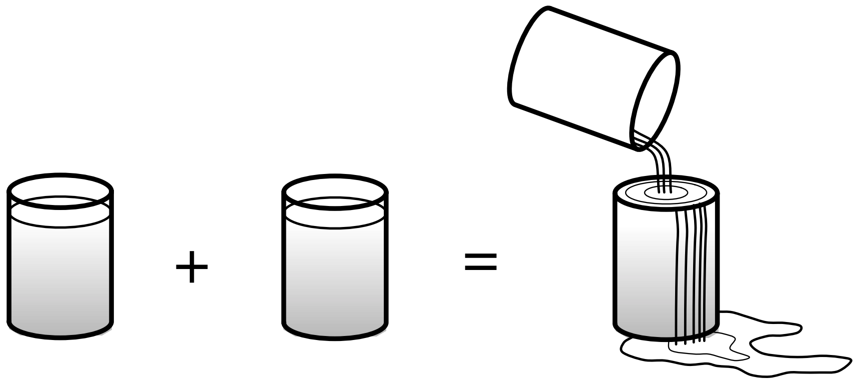

Figure 2.11c: For full cups, 1 + 1 = 1 + 0

Flipping the light switch plus flipping the light switch again equals no change. Doing the same thing twice can leave you where you started. Actions can be added but sometimes adding subtracts (Figure 2.11d).

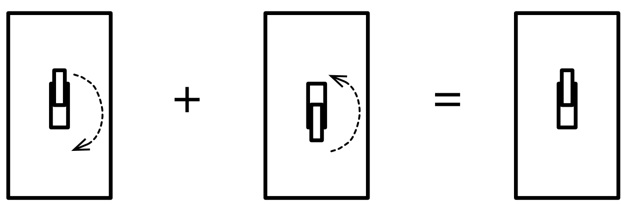

Figure 2.11d: For flipping light-switches, 1 + 1 = 0

Nine o'clock plus four hours equals one o'clock. For clocks, when you count up to twelve, the next number is one. Clock arithmetic is cyclic, how high we can count is limited (Figure 2.11e).

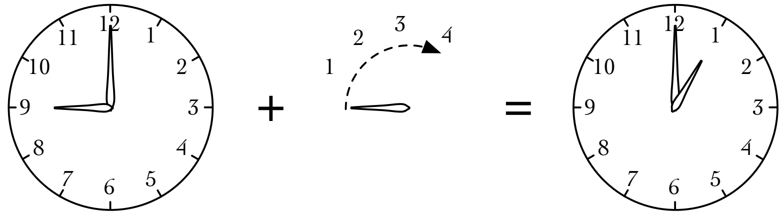

Figure 2.11e: For clock time, 9 + 4 = 1

A pile of beans plus one bean is still a pile of beans. For piles of beans and for crowds of people, one more does not change the total. You can add to a large collection without changing the size of the collection (Figure 2.11f).

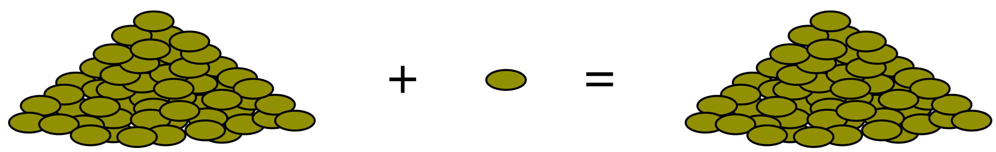

Figure 2.11f: For large groups, many + 1 = many

It is tempting to say that these are exceptions, that these kinds of addition are not really addition. Yes, they are not school addition but neither are they exceptions within our real world. Each is authentic, an exception only inside a classroom.

Unreasonable Addition

A common exercise for students in fourth grade is to add two abstract four digit numbers:

$$\begin{array}{r} 2639 \\ +\ 1143 \\ \hline \end{array}$$

These numbers are not authentic because they do not refer to any actual things. Worse, there are no actual circumstances that might legitimize these numbers, simply because there is no authentic occasion in which fourth graders (or anyone else) will count exactly up to 3782. Schools do not attempt to make this exercise real, students simply follow memorized rules blindly, as practice in manipulating the code, without purpose, without motivation, without reason. Pity our poor students, they are made to repeat this meaningless activity hundreds, if not thousands, of times. They are indoctrinated until they conform. They are forced to abandon their knowledge of physical reality in favor of the catechism of exact numbers.

Often textbooks will wrap these addition exercises in deceptive words, hiding meaningless math within unreasonable stories. *Word problems* embed manipulation tasks in non-authentic applications. Most students hate word problems. Math textbooks deceive when they describe word problems as "Math with Applications". What they should probably say is "Abstract Math with Words Used to Create an Illusion of Authenticity". Here is an example:

> "Sara has $26.39 in her piggy bank and her mother gives her $11.43 for her birthday. How much does Sara have?"

No! This is a thinly veiled attempt to get the student to add large numbers exactly. The authentic answer depends on what Sara wants to know about her money (Figure 2.12). Probably something like, "I have close to $40." Or, "I have enough to buy those shoes." The exact answer, $37.82, is exact only when Sara's wealth is unchanging and unambiguous. The exact answer holds only until Sara acquires another penny. The exact sum is fleeting, unstable information. This is not to mention the unlikelihood of receiving exactly $11.43 from your mother on your birthday. And not to mention the unlikelihood that Sara has actually taken the time to count, exactly, the wealth within her piggy bank.

Figure 2.12: Birthdays are not math problems.

An authentic objective might be to understand Sara's monetary situation, to estimate what she has in total. However the skill of estimation is not taught, rather it is punished. An answer of "about $40", even an answer of $37.80 (ignoring the odd penny or two), is WRONG when an exact counting answer becomes more important than a sensible answer. Word problems miss the point of authentic addition by failing to teach the essential skill of reasoning about quantity. Unreasonable addition is justified as "practice with the manipulation of numbers". The theory is that practice with essentially random numbers will help a person to add some other numbers in some other context. This theory has two problems. Skills anchored to random information are not retained in memory, we forget skills that are not relevant. More importantly, the general skill itself is no longer relevant in today's technological society. If it were important for Sara to know the exact sum, then it would be irresponsible for her not to use a calculator. If it were not important to know the exact sum, then it would be naive to expect her to add exactly.

Commutativity, Not

Most of us have been exposed to the rules of addition. Consider one of them, commutativity. For our current example,

$$2639 + 1143 = 1143 + 2639$$

Commutativity says that when you add two numbers, the result is the same no matter which one is added to the other. Here is the problem: it is not necessary to choose either to be first. There is no actual ordering when two authentic numbers are added together. That is what commutativity is saying, but it is *not* a rule, it is an obvious consequence of physical space.

Were we to put a pile of exactly 2639 of something (something small I hope, perhaps beans or if you are as fortunate as Sara, pennies) on a table, and another pile of exactly 1143 of the same kind of thing on the same table, and then invite some kids to "add them together", what do you think would happen? Would the kids carefully select one of the piles, and then add the other pile to it, or might they simply push both piles together at the same time (Figure 2.13)? The physical definition of addition is to join together. This definition says nothing about a first and a second thing. Authentic adding occurs in physical space, *all at the same time*. Reality supports parallelism, many things (thank goodness) can happen all at once. Not so for symbolic strings. That is, the rule of commutativity is not about adding, it is about how strings of symbols work. People don't add symbols in the real world, we add things. There is certainly a first number in the symbolic representation of numbers, but this firstness is not authentic, it is disconnected from what is actually real.

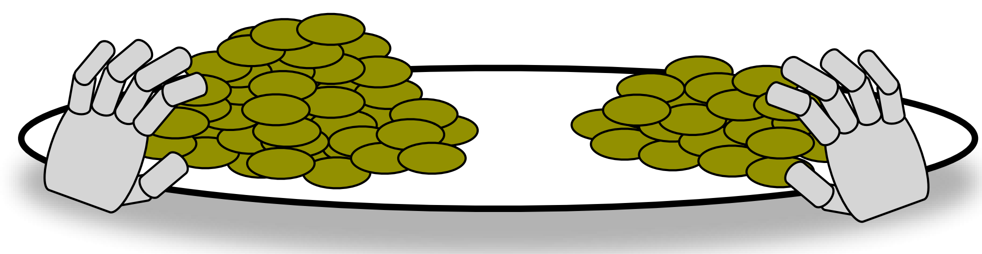

Figure 2.13: To add two things, push them together.

Authentic addition, the kind done by people in a physical space, has nothing to do with commutativity, or with associativity for that matter. The rule of *associativity* says that to add three piles of things, we must select two of them (be sure that one is first), add them, and then add the third. Associativity says that it doesn't matter which pile we choose as first or second or third. Yes! Absolutely correct, it does not matter. It does not matter because sequential addition is not about addition at all, it is about symbol manipulation. Try the pile test on some kids. Put say five piles of beans on a table, and ask the kids to add these piles together (Figure 2.14). Most will think that you are asking for a mental feat, an exact counting. Assure them that you do not care about counting how many, that you just want to add the piles together. Only the most indoctrinated will push the piles together one at a time.

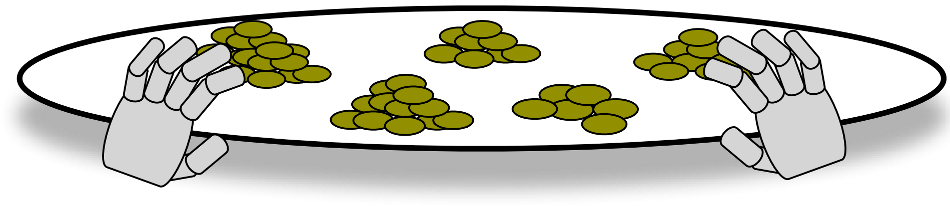

Figure 2.14: To add many things, push them all together.

Unit ensemble arithmetic shows that we can understand addition, and the rest of the operations of arithmetic, without separating theory from practice. This is a form of arithmetic that is not disembodied, a form that makes physical and visceral sense. It also meets the rigorous standards of formal mathematics, providing an alternative that escapes the theology of infinity and the discorporation of the body without abandoning the comfort of proof.

Authentic Geometry

Counting and adding arose naturally at the beginning of commerce around eight to ten thousand years ago. Geometry, however, was invented by the ancient Greeks twenty-five hundred years ago. The ancient Babylonians and Egyptians knew how to subdivide land and build pyramids, but the Greeks freed these pragmatic skills from their concrete applications and placed them within an abstracted, formalized mathematical system.

Perfectly Greek

Euclid's *Elements* [8] introduced the axiomatic style into mathematics by postulating perfect points and perfect lines and perfect shapes, all laid out on a perfectly flat plane. This form of thinking dominated geometric and mathematical thought for two millennia. The problem is that extentless points and straight lines and flat planes are all imaginary constructs. It is particularly difficult not to see Greek geometry as natural, since it dominates our cultural heritage. Nothing seems more natural than a triangle or a square or a circle. But each of these figures inhabits a perfect Platonic virtuality that the Greeks also invented. Perfect shapes are not realizable, so the Greeks provided an unreality to contain them.

We know the value of π, for example, to a trillion digits (!) [9], but what *is* π? From a simple perspective, π is the ratio of the radius of a perfect circle to its circumference. The billions of digits of π require a perfect circle. To connect this theoretical accuracy to an authentic situation, it is necessary to measure the radius of a perfect circle perfectly, or at least to an accuracy of a trillion digits. If we manage to get an authentic measurement of a radius to twenty decimal places, then the value of the circumference cannot be known to more than twenty decimal places. All those billions of digits of π past the first twenty are lost into irrelevance. The same fate awaits any large number generated by the purely theoretical constructions of a perfect geometry. We can know a trillion decimals of π only when they are not participant in any kind of authentic reality (Figure 2.15).

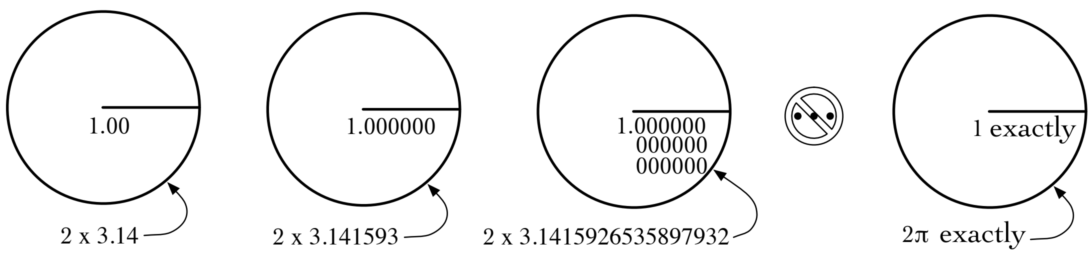

Figure 2.15: An authentic circle is only as perfect as its measurement.

The perfect square has four equal sides. Building a perfect square would require four different measurements (one for each side), all in perfect agreement. Greek geometry studied these abstractions *without measurement* because Greek geometry did not include the concept of measuring. It could not, of course, because then it would not have been perfect. Perfect and authentic do not coexist. The fantasy of Euclidean geometry has proved to be invaluable for construction of human habitat. We just love flat floors and roofs and desks. But Euclidean geometry does not occur in the natural world. An authentic geometry would anchor to our bodies, to our senses, and to our experiences. Authenticity is not necessarily a consequence of what we construct, it must also be a consequence of Nature. Alas, there are no extentless straight lines and flat surfaces in Nature.

As Einstein showed us, empty space is incredibly complex, it is inseparable from time and it is curved by the presence of mass. Einstein's spacetime seems to be more authentic than Euclid's perfect flatness. But the geometry we are looking for is not cosmological, since authenticity also requires a kind of naturalness, an ease of human understanding. Authenticity does not negate advanced mathematics, the kind of math that very few people understand, the kind that requires years of intense study. Authentic math seeks only to carve out a portion of the discipline of mathematics and return it to common understanding. We know that Nature is incredibly complex, the challenge is to find those aspects of mathematics that align both with Nature (also known as reality) and with human intuition. The kind of math that allows most people to say, "Yes, of course".

Fractal Monsters

Benoit Mandelbrot published his research on fractal geometry in 1977. *Fractals* are deeply mathematical and still not well understood, but they are a candidate for an authentic geometry of Nature. The mathematical forms Mandelbrot studied were known a hundred years before, when they were called "monsters". They were infinitely complex and infinitely unruly. Unbounded unruliness simply means that all of the comfortable rules of Euclidean geometry as well as those of calculus do not work for these fractal monsters. Fractals are truly a child of computational mathematics, they cannot be explored without computers. They are geometric forms that are better understood by observation than by formula. But their astounding strength is that when we look at them, they bear a distinct resemblance to what we observe in Nature. This visual alignment has been explored deeply in movies made using computer graphics, movies that show us visually credible trees and mountains and clouds that were constructed using fractal techniques. The process that generates fractals has also been called upon as an explanation of phenomena as diverse as biological growth, the spread of disease, the creation of rivers, and the measurement of carbon sequestration in forests (Figure 2.16).

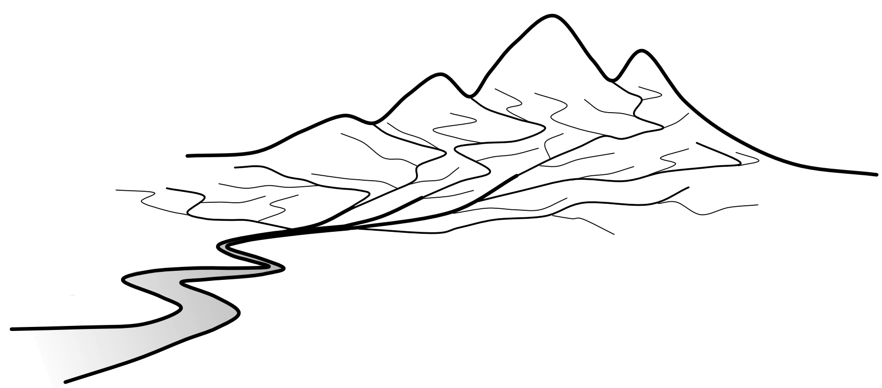

Figure 2.16: The fractal growth of a river.

The two central ideas of fractals are self-similarity and iterated growth. *Self-similarity* means that when we look at different scales of magnification, we see the same thing. The meandering of a river as it enters its delta is replicated at a smaller scale by the tributaries coming together to form that river. The structure of the tributaries is replicated by streams at a smaller scale. Streams are replicated at a smaller scale still by rivulets of collecting raindrops. The important idea is that at each scale, the same thing seems to be occurring, and the identification of sameness is visual, apparent. *Iterated growth* is a way to construct self-similarity by feeding the same simple pattern back into a growing form. Fractals explain the complexity we see in natural systems as simple patterns replicated a multitude of times at different scales. Complexity is *lots of simplicity*. A tree is a complicated organism, how does it know how to grow? The fractal geometry of trees suggests that the budding tip of a growing tree does not need to have the entire evolution of the tree encoded in its stem cells. Cells turn into buds by the same process that buds turn into twigs, twigs turn into branches, and branches turn into trunks. Like the river, a tree grows by repeating a simple process many times at different scales of organization.

Figure 2.17: The fractal growth of a bush.

The above fractal schematic for a bush begins with three straight lines (Figure 2.17). This pattern is reduced to half size and reinserted into itself many times, once for each extant line. The second pattern is then shrunk and again reinserted, producing an outline of the growing bush. The pattern is again shrunk and reinserted to generate the fourth image, a mature bush.

How does a cloud form? Microscopic drops of water condense to form a wisp, wisps form into patches, patches into cloudlets, cloudlets into clouds, clouds into storms. The same simple process repeated many times at different scales of organization. Fractal concepts are simple, but due to the dominance of Euclidean idealism in our culture, fractal concepts are as yet still unfamiliar. Once understood, not mathematically but visually and experientially, they appear to be everywhere. Tiny waves of seafoam accumulate on the shoreline, driven by wavelets of the same seastuff, driven by larger waves pounding the shoreline, driven by still larger swells crossing an ocean. Minute plankton become food for tiny fish that are eaten by small fish that are eaten by larger fish, and so on up the food chain, each participant performing the same act of feeding on smaller creatures, each at a larger scale. Repetition of the simple act of eating a smaller fry cascades to create the complex web of ocean life, fractally.

Authentic Logic

Historically, logic has been associated with the way that our minds work. It has been with us in one form or another for 2500 years. Logic is supposed to be essential to rational thinking, but we know for certain that it is generally incomprehensible to students. *Elementary logic* is about propositions, statements that can be determined to be either True or False. Yes, it's true, logic describes how to think in black-and-white, without nuance, without context, without a sense of self. Like arithmetic, at the beginning of the twentieth century, logic was constructed to be devoid of authenticity. Logic became symbolic. The gain was that truth and proof found a firmer ground. This *symbolic logic* became fundamental to computer operations, it serves marvelously to organize wires and transistors and pixellated displays. Today's logic defines how computers are designed and how they operate, but it no longer helps people to think clearly (Figure 2.18). Common sense logic has been sacrificed to virtuality.

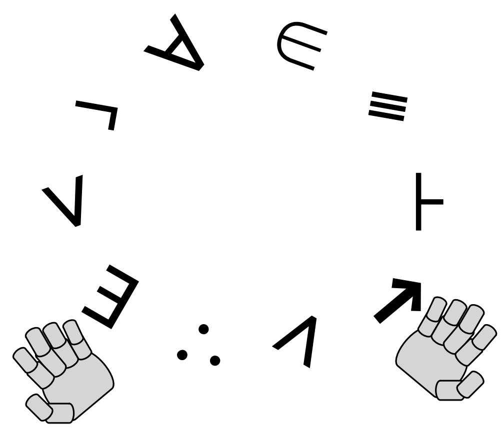

Figure 2.18: Symbol juggling does not lead to clear thinking.

Logic is a system of values, a way of looking at the world, it's a belief system that has matured into an algebra. *Mathematical logic* includes many of the same concepts as the algebra of numbers, concepts like associativity and commutativity. In identifying authentic addition, these concepts melted away; they are also inessential to authentic logic, to logic that helps us to think clearly. Unlike the arithmetic of numbers, we cannot draw upon historical precedence to return logic to a humane foundation. Unlike arithmetic, logic has never been anchored to physical reality. It is instead a creation of language. How then can logic be made authentic? We need to stop confusing clear thinking with the juggling of meaningless symbols. For this, it's essential to modify the current symbolic form, to make logic iconic. We can also remove the redundancy: logic is about one type of relation, not two, not sixteen. Most importantly, we can reconnect the discipline of logic with the human body. These steps can insure that we apply logical thinking to reality rather than to virtuality.

There *is* an authentic logic, a logic that we can manipulate with our hands, a logic that we can understand with our bodies. This kind of logic, invented in the late nineteenth century, is equally powerful in producing clear, correct thoughts. During the formalization of logic, pioneers such as Venn and Peirce developed formal iconic versions of logic. These visual forms were expunged in the rush to symbolization, but they provide a method to deliver logic back to common comprehension. Extending C.S. Peirce's astonishing invention of Existential Graphs [10] a hundred years ago, we can construct a humane logic that can be manipulated with our eyes and with our hands. Peirce's invention makes it possible to design an iconic logic that illustrates what it means. Elementary logic can be expressed completely by the distinction between the inside and the outside of a logical container. That A implies B means that A is on the inside while B is on the outside: (A) B. Certainly a different way of thinking, but one that is necessary to escape the mire of convoluted thinking that currently passes as logic. From the basis of logical containers, logic takes on a different metaphysics. It is no longer about the polar extremes of True and False, it is simply about structured distinctions. Surprisingly, when Peirce pursued this line of thought, he found that the rules and operations of logic reduce to construction and deletion of containers. Elsewhere I've called this approach Boundary Logic [11], the simplification of symbolic logic into distinctions formed by iconic boundaries (Figure 2.19).

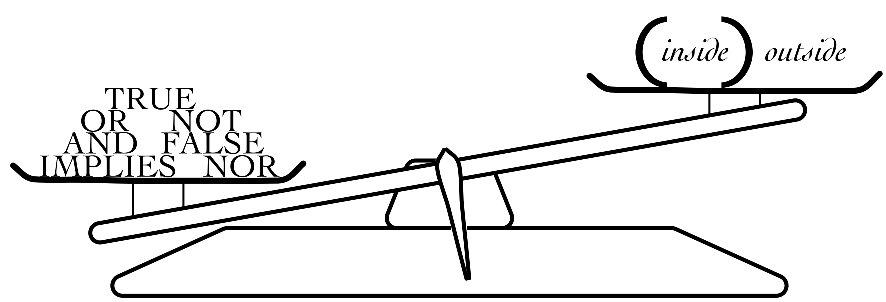

Figure 2.19: Logic is heavy with unnecessary concepts.

Here then is a perspective for *humane logic*: logic is about the structure of ideas, not about the putative Truth of the ideas themselves. Iconic logic is a method of creating an ecology of thought within which essential ideas are retained while inessential ideas are discarded. Deduction occurs via deletion of irrelevance, an approach that is significantly more clarifying than proof via the accumulation of facts. Symbolic logic can be made more humane by tossing the accumulation of conceptual baggage it carries into a void, into nothingness. What remains is what matters. And like all symbolic mathematics, logic benefits greatly in clarity and comprehension by getting rid of the meaningless symbols.

Many different types of mathematics are firmly anchored to meaning. The rotation group of a cube describes how objects move in space. Rotation is authentic, it is cubes that are not. My thesis is that we need to distinguish between these types of math, teaching some of them to young children, providing some as speciality tools to scientific disciplines, burying some of them deep within microscopic folds of silicon, offering some only to declared professionals in the field of mathematics, any protecting most of us from most of them. We next face the villain squarely: what type of math is *school math*?

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